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PRINCIPLES OF MECHANISM

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PREFACE

The modern trend toward automatic operation and higher speeds in all types of machinery and heat engines has made the analysis of the motions of machine and engine parts essential to good design.

For a number of years the authors have been associated with students taking courses in mechanism, or kinematics, and this book is largely a compilation of the material used in such courses. We have intended this work primarily as a textbook for engineering students, and have developed it with the idea of correlating the work usually required of engineering students in mathematics and physics with the subject matter of the text, at the same time laying a foundation for the advanced courses in machine design which usually follow this course.

We recognize the time limitation usually imposed on this course in the curricula and have left out a great deal of the descriptive matter frequently found in texts of this type; nevertheless, the book contains adequate material and is sufficiently flexible to be suitable for courses of from two to four semester hours. For the shorter courses certain chapters may be omitted without destroying the continuity. We have made a special effort, however, to develop the fundamentals of the subject in a logical order so that the initiative and reasoning power of the student will progress with the text.

Having found that students beginning this course do not have a good understanding of vector quantities, we have included a chapter on vectors, sufficiently developed to give the student an understanding of their application to the analysis of displacement, velocity, and acceleration problems.

Several methods of solving the same type of problem are used to impress upon the student the advantage of attacking a new problem from different points of view. Examples and illustrative problems are worked out in detail in the early chapters, but this detail is gradually abandoned as the subject is developed. This procedure gives the student a decreasing amount of help in applying the principles acquired and at the same time facilitates review. Numerous problems are included at the end of each chapter throughout the book, all of which are suitable for class or home work, while some are suitable for work in the drafting room.

Material taken from various sources is acknowledged in the text but we wish here also to acknowledge the kindness of the manufacturers who have furnished cuts and illustrations.

A. Vallance
M. E. Farris

April, 1933.

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PRINCIPLES OF MECHANISM

CHAPTER I

INTRODUCTION

1. Mechanisms. A machine may be considered to be a combination of fixed and moving parts (such as links, wheels, gears, belts, and similar devices) combined to transmit forces, power, and motion from a source of power, in order to secure some definite useful form of work. Any combination of two or more parts of the machine mechanically connected so that they have definite motions with respect to each other, is called a ***mechanism.***

In the design of a machine, the motions of the various parts and the forces transmitted must be considered as well as the size, shape, and strength of the parts. The study of the motions and forces transmitted, regardless of the strength of the parts, is called the ***science of mechanism.*** The determination of the size and shape of the parts so that they may safely transmit the forces impressed upon them belongs to another branch of the subject of machine design, and will not be considered in this work.

A mechanism as defined above is often called an ***elementary mechanism,*** and may consist of

(*a*) Connected rotating, reciprocating, or oscillating bars, rods, or links.

(*b*) Revolving shafts, connected by friction wheels, gears, or cams.

(*c*) Flexible connectors such as belts, ropes, or chains, with the necessary pulleys, sheaves or sprocket wheels.

(*d*) Fluids (liquids and gases) with their restraining cylinders, plungers, or other elementary mechanisms used to transmit motion or power, as in hydraulic presses.

A ***train of mechanism*** is a combination of two or more elementary mechanisms.

A ***driver*** is the part of a mechanism which imparts motion to a second part or follower.

A ***frame*** is the portion of the machine or mechanism which serves to support the various operative parts and maintain them in their proper relative positions. The frame is not necessarily a stationary part; consider, for example, the frame of an automobile or locomotive.

A ***link*** is a rigid piece connecting two moving parts of a mechanism.

A ***band*** is a connecting piece which is not rigid, but is flexible and can transmit motion only by a pull.

A ***cycle*** is a definite repeating sequence of motions of the parts of a mechanism. During the operation of a machine or mechanism, the parts go through periodic changes of position, a complete series of which is a cycle. After completing a cycle, the parts of the mechanism have returned to their original relative positions.

Velocity is the time rate of motion with which any particle moves through space.

Absolute velocity is the velocity of any particle with reference to some fixed or stationary particle. In the science of mechanism, points on the earth are usually considered to be stationary reference points.

Relative velocity is the velocity of any particle with reference to some other particle which may itself be moving.

Linear speed is the time rate of motion of a particle at any particular time, and does not include direction of motion as velocity does.

Velocities and speeds are measured in feet per second, feet per minute, and miles per hour.

Angular velocity is the time rate of motion with which a body turns or rotates about an axis. It may also be defined as the time rate with which any line on the body changes direction.

Angular velocities are measured in revolutions per minute, degrees per minute, and radians per second.

Uniform speed: When equal intervals of space are traversed in equal intervals of time, the speed is uniform.

Variable speed: When unequal intervals of space are traversed in equal intervals of time, the speed is variable.

Acceleration is the time rate of change of velocity. Acceleration may be linear, as when the linear speed changes; or angular, as when the angular speed changes. Since velocity denotes direction as well as linear speed, there may be a change in linear speed, a change in direction, or a change in both. If there is a change in the linear speed in the direction tangent to the path over which the particle is travelling, this change of speed is called ***tangential acceleration.*** If the direction of motion is also changing, there is an acceleration perpendicular to the path called ***normal*** or ***radial acceleration.*** The true acceleration is the vector sum of the tangential and normal accelerations. If the rotation of a body is uniform, the tangential acceleration of any particle in the body is zero, but there is a normal acceleration towards the axis of rotation. If the body has angular acceleration, any particle has both tangential and normal acceleration.

When the velocity is changing at a constant rate, the particle is said to have ***uniform acceleration.*** When the velocity is changing at a variable rate, the particle is said to have non-uniform acceleration. Acceleration may be positive or negative, depending on whether the velocity is increasing or decreasing. Negative acceleration is called deceleration, or retardation.

2. Speed and Acceleration Relations. Throughout this text the following symbols will be used to express the relations between space, time, speed, and acceleration:

s = space traversed, feet.
θ = total angle through which the body rotates, radians.
t = time interval, seconds.
v = linear velocity, feet per second.
ω = angular velocity, radians per second.
a = linear acceleration, feet per second per second.
a_t = tangential acceleration, feet per second per second.
a_n = normal acceleration, feet per second per second.
α = angular acceleration, radians per second per second.

r = distance of any particle of the body from the axis of rotation, feet.
N = revolutions per minute.

UNIFORM VELOCITY

	Linear	Angular

Distance = (Velocity) × (time)

	Linear	Angular
(1)	$s = vt,$	$\theta = \omega t.$

Since there is no change in velocity the linear or tangential acceleration will be zero.

	Linear	Angular
	$a = a_t = 0,$ (no change in vel.)	$\alpha = 0,$
(2)	$v = \frac{2\pi r N}{60} = \omega r,$	$\omega = \frac{2\pi N}{60} = \frac{v}{r},$
	$\boxed{a_n = \frac{v^2}{r}}$	$a_n = r\omega^2.$

VARIABLE VELOCITY—UNIFORM ACCELERATION

	Linear	Angular

Final velocity = Initial velocity + (Acceleration) × (time)

	Linear	Angular
(3)	$v_2 = v_1 + at,$	$\omega_2 = \omega_1 + \alpha t.$

Distance = (Initial velocity) × (time) + (1/2)(Acceleration) × (time)²

	Linear	Angular
(4)	$s = v_1 t + \frac{1}{2}at^2,$	$\theta = \omega_1 t + \frac{1}{2}\alpha t^2.$

(Final velocity)² = (Initial velocity)² + 2(Acceleration) × (distance)

	Linear	Angular
(5)	$v_2^2 = v_1^2 + 2as,$	$\omega_2^2 = \omega_1^2 + 2\alpha\theta.$

VELOCITY IN GENERAL

	Linear	Angular
(6)	$s = f(t),$	$\theta = f(t),$
(7)	$v = \frac{ds}{dt},$	$\omega = \frac{d\theta}{dt},$
(8)	$a = \frac{dv}{dt} = \frac{d^2s}{dt^2},$	$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2},$

(9) $a\,ds = v\,dv,$ $\alpha\,d\theta = \omega\,d\omega,$

(10) $\alpha = \dfrac{a_t}{r},$

(11) $\omega^2 = \dfrac{a_n}{r}.$

PROBLEMS

1. A train travels 2 hrs. and 30 min. and goes 125 mi. Express its average speed in mi. per hr., ft. per min., and ft. per sec.

2. A body starting from rest has a uniform acceleration of 10 ft. per sec. per sec.
   (a) What is the velocity at the end of 5 sec.?
   (b) How far has it moved in 5 sec.?

3. A body falls freely for 3 sec., starting from rest.
   (a) What is its velocity after 1, 2, and 3 sec. have elapsed?
   (b) What is its acceleration after 0, 1, 2, and 3 sec.?
   (c) What space is traversed in the first 3 sec.?

4. An automobile accelerates uniformly from 4 mi. per hr. to 30 mi. per hr. in 20 sec.
   (a) What is the rate of acceleration in ft. per sec. per sec.?
   (b) How many ft. are traversed in the 20 sec.?

5. A body moves according to the equation $s = 2at^3$. The body starts from rest and after 2 sec. has moved 32 ft.
   (a) Determine the acceleration.
   (b) Determine the velocity at the end of 2, 3, and 4 sec.

6. A body traveling at the rate of 120 ft. per min. has a constant force applied to it, which causes it to reach a speed of 240 ft. per min. after traveling 50 ft. What is the acceleration in ft. per sec. per sec.?

7. A bullet leaving the muzzle of a gun 36 in. long, has a velocity of 1500 ft. per sec. If this velocity was acquired with uniform acceleration, determine the acceleration and the time required to reach the muzzle.

8. A body moves along a straight path 5 ft. long in 10 sec. It starts from rest, and during the first 3 sec. it moves with uniform acceleration, during the next 4 sec. it moves with uniform velocity, and during the last 3 sec. it has uniform deceleration and comes to rest.
   (a) What is the acceleration in ft. per sec. per sec.?
   (b) What is the maximum velocity in ft. per sec.?
   (c) What space is traversed at uniform velocity?

9. A wheel is rotating with an angular velocity of 5 radians per sec. How many revolutions will it make in $1\frac{1}{2}$ min.?

10. A centrifugal-pump impellor revolves at the rate of 1750 r.p.m. What is its angular velocity in radians per sec.?

11. A factory line-shaft revolves at 250 r.p.m. What is the angular velocity of a point on the rim of a 30-in pulley carried on this shaft in radians per sec.?

12. The length of stroke of the table of a planer is 4 ft. The cutting stroke is made in 9 sec., and the return stroke in 6 sec. What is the average cutting speed in ft. per min.? What is the average return speed? What is the average speed during a complete cycle?

13. A 24-in. pulley on a factory line-shaft revolves at the rate of $2\frac{1}{2}$ radians per sec. How many revolutions will it make during an 8-hr. working day?

14. A body traveling with a linear speed of 100 ft. per min. was accelerated for $\frac{1}{2}$ min., after which its speed became 1000 ft. per min. If the acceleration was uniform, what was its value in ft. per sec. per sec.? How far did the body travel during the accelerating period?

15. A 3-ft. wheel rolling on a street was stopped with uniform retardation in 48 ft. What was the angular retardation in radians per sec. per sec. and in revolutions per min. per min.?

16. A fly-wheel is rotating at 300 r.p.m. and if allowed to revolve freely friction will bring it to rest in 75 sec. Determine the angular retardation in radians per sec. per sec.

17. A steam turbine whose rotor is 30 in. in diameter is brought up to full speed of 1800 r.p.m. in $1\frac{1}{2}$ min.

(a) Find the average acceleration in radians per sec. per sec.
(b) How many revolutions did the rotor make before coming to full speed?

18. A fly-wheel 15 ft. in diameter is rotating at 80 r.p.m. A constant force is applied at the rim which brings the wheel to rest in 3 min.

(a) What was the deceleration in radians per sec. per sec.?
(b) How many revolutions did the fly-wheel make before stopping?

19. A wheel 6 ft. in diameter is increasing its speed at the rate of 120 r.p.m. per minute. At the end of 15 sec., what is the angular speed? What is the linear speed of a point on the circumference? What is the tangential acceleration of this point? What is the normal acceleration of this point?

20. A $\frac{1}{2}$-lb. roller $\frac{3}{4}$ in. in diameter rolls on the inside of a ring 6 in. in diameter, whose axis is horizontal. How fast must the roller move so that it will not fall from the ring when at the highest point in its path?

CHAPTER II

VECTORS

3. Scalar and Vector Quantities. When quantities are dealt with, two kinds must be considered. First, quantities involving the idea of magnitude only, such as weights, volumes, areas, amounts of energy, etc., are called ***scalar quantities.*** Second, quantities involving the idea of direction as well as magnitude, such as forces, accelerations, velocities, etc., are called ***vector quantities.*** Magnitude alone cannot define vector quantities, but their direction and sense must also be known. For example, if a force of 100 pounds acts on a body, the effect of the force cannot be realized unless one knows whether the action of the force is vertical, horizontal, or inclined; whether its action is up or down, to the right or to the left; and at what point in the body the force is applied.

Scalar quantities of the same kind can be added and subtracted by simple arithmetic, but vector quantities cannot be so readily combined. It is the purpose of this chapter to show some of the methods of representing vector quantities, and some of the methods of combining them.

4. Vectors. Any vector quantity may be represented by a straight line with an arrowhead placed at one end. The length of the line, to scale, represents the magnitude of the quantity. The direction is indicated by the slope of the line. The sense is indicated by the arrowhead. The position of the vector in space is immaterial, since it does not indicate the position, or point of application, of the vector quantity.

In Fig. 1, the line AB represents a vector quantity, acting upward to the right, at an angle of 30 degrees with the horizontal. To scale, the length of the line indicates the magnitude of the quantity. The slope (30 degrees above

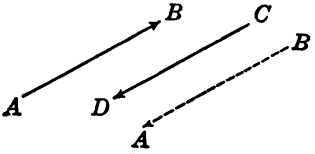

FIG. 1

the horizontal) indicates the direction. The arrowhead indicates the sense, that is, the quantity acts away from *A* and toward *B*. The vector *CD* represents a quantity of the same magnitude and direction, but of opposite sense.

If sense from *A* to *B* is arbitrarily said to be plus, then $+ AB$ represents the vector quantity. If the sense is reversed, then $- AB$ or $+ BA$ represents the vector quantity. The reversal of sense is indicated by reversing the arrowhead, as shown by the dashed lines in the figure.

The tail of the arrow representing a vector quantity is called the ***origin,*** and the head of the arrow is called the ***terminus.***

5. Vector Addition. By following certain rules, one can perform the processes of addition, subtraction, multiplication, etc., on vector quantities. This cannot be done by arithmetical addition, subtraction, or multiplication, because the process must take into consideration the direction and sense of each of the vector quantities. Addition and subtraction are the only processes of interest in this work on mechanism.

The process of addition is shown in Fig. 2. The vectors *a* and *b* are placed together so that the origin of vector *b* coincides with the terminus of vector *a* at *B*. A vector *AC* joining the origin of the vector *a* and the terminus of the vector *b* is their vector sum, or ***resultant.*** The vectors *a* and *b* are called ***components*** of the vector *AC*.

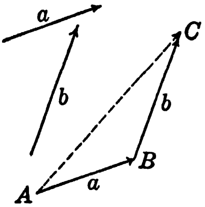

FIG. 2

EXAMPLE. Suppose that a man walks from *A* to *B* in a straight path. This motion or displacement is fully indicated by the vector *AB*, Fig. 2. At the point *B* the man turns and walks in a straight line to *C*. This motion is indicated by the vector *BC*. The net result of the walking is that the man has traveled from *A* to *C*. The same result could be accomplished if he walked in a straight path directly from *A* to *C*. The vector *AC* indicates (in magnitude, direction, and sense) the net displacement accomplished, or the net result of the walk from *A* to *B* and the walk from *B* to *C*.

In this example each vector represents displacement or space traversed. Since velocity is space traversed, divided by the time interval, it is evident that the vectors may also represent velocities as well as displacements.

6. Vector Polygon.* Two, three, or any number of vectors may be added by following a simple rule for addition. Starting with any of the vectors to be added, draw the second vector with its origin at the terminus of the first, draw the third vector with its origin at the terminus of the second, etc., until all the vectors have been drawn. The line drawn from the origin of the first vector to the terminus of the last vector is the vector sum or resultant. The complete figure is called a ***vector polygon.***

The order in which the vectors are combined is immaterial, so long as the arrowheads point to form a continuous path. In Fig. 3, five vectors a, b, c, d, and e are combined in several ways, the resultant vector R being the same in every case.

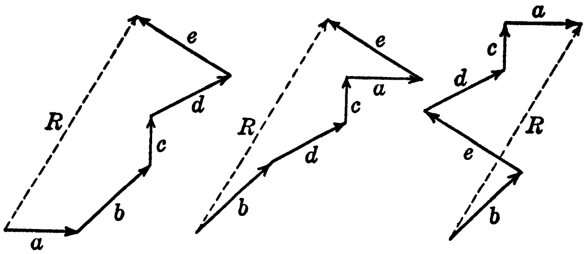

FIG. 3

When vectors are added, it may happen that the terminus of the last vector coincides with the origin of the first vector. In this case, the vector sum is zero, the resultant being zero. If the vectors combined are force vectors, the vector sum may be zero, while the resultant is two equal and opposite forces in the form of a "couple."

A vector addition may be indicated as

$$\text{vector sum } (AB + BC + CD) = AD, \tag{12}$$

---

* A discussion of space vectors has not been included in this text because the subject matter dealt with involves only coplanar vectors. The student will learn various methods of dealing with space vectors in his study of applied mechanics.

or

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}.$$

7. Vector Components. Two or more vectors can be added to give a resultant vector or sum. Obviously, the resultant can be broken up into the original vectors. Any vector quantity can be broken up and replaced by a number of vector components whose sum is equal to the original vector.

In Fig. 4, the vector *AB* is broken up into the vectors *AC* and *CB*, and also into three vectors *AD*, *DE*, and *EB*. It is obvious that there is an indefinite number of combinations of two or more components into which any vector quantity can be broken up, and that these combinations are equivalent to each other, since their sum is always the same. In most cases where it is desired to resolve a vector into components, the number and directions of the desired components are known in advance, but their magnitudes are not.

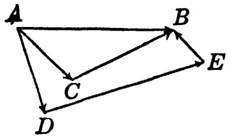

FIG. 4

In dealing with vectors representing forces, it is usually desirable to resolve the vectors into two components acting parallel to two reference axes, and perpendicular to each other. The reference axes usually chosen are the horizontal and vertical, or the *X* and *Y* axes. The components are then called the *X* and *Y* components.

EXAMPLE. In Fig. 5, a rope 20 feet long is suspended from two points 15 feet apart on the same horizontal line. A weight of 75 pounds is hung from the rope at a point 8 feet from one end. Disregarding the weight of the rope, find the force acting on each section of the rope.

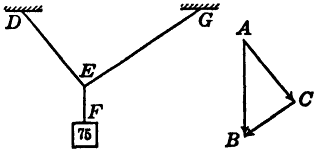

FIG. 5

The forces in the two sections of the rope must be such that their vector sum is equal to the weight, and the directions of the forces must be along the axes of the ropes. These forces may be found by resolving the known 75-pound force into two components whose directions are also known. Draw the vector AB representing, to scale, the 75-pound force. From A draw a line, AC, parallel to the section of rope DE. From B draw BC parallel to the section of rope EG. The lines AC and CB are vector components of the vector AB, and to scale represent the forces pulling on the two rope sections. By measurement, the force on the rope DE is found to be 65 pounds, and the force on the rope GE is found to be 47 pounds.

8. Vector Subtraction. In Fig. 2, two vectors, AB and BC, have been added, the sum being the vector AC. If BC added to AB gives AC, then BC subtracted from AC must give AB. Thus, if the vector sum $(AB + BC) = AC$, then

$$\text{(13)} \qquad \text{vector difference } (AC - BC) = AB = -BA,$$

and

$$\text{(14)} \qquad \text{vector difference } (AC - AB) = BC = -CB.$$

Vectors may be subtracted according to this simple rule. *Draw both vectors from the same point or origin, with the arrowheads pointing away from this point. The line joining the termini of the two vectors is their difference.* The sense of this vector is found by placing the arrowhead pointing toward the vector from which the subtraction is made.

In Fig. 6, the vector AC is subtracted from the vector AB. The vector difference $(AB - AC) = CB$. The sense of the vector representing the difference is toward B.

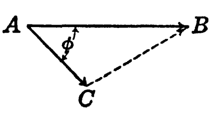

FIG. 6

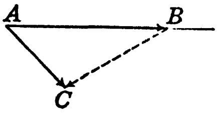

FIG. 7

In Fig. 7, the vector AB is subtracted from the vector AC. The vector difference $(AC - AB) = BC$. The sense of the vector representing the difference is toward C. Note that the magnitudes of the two vectors in Figs. 6 and 7 representing the differences are equal, but the senses are opposite.

In § 7, it is shown that any vector quantity may be replaced by two or more components. It follows that it is possible to subtract one, two, or more vectors from a single vector, or from the sum of several vectors.

9. Relative Quantities. In the study of mechanism the value of the displacement, velocity, or acceleration of one body relative to some point or to some other body is of great importance. Displacements, velocities, or accelerations are called ***absolute*** or ***relative,*** depending on whether the reference point or body is fixed or moving. Since mechanisms operate on or near the surface of the earth, the earth is considered to be a fixed body in the analysis of mechanisms, so that any point on the earth may be taken as a reference point, and its own displacement, velocity, or acceleration in the universe may be disregarded, or considered to be zero.

The displacement, velocity, or acceleration of any particle in a mechanism, with reference to any point on the earth, is called an ***absolute quantity.*** The displacement, velocity, or acceleration of any moving particle on the earth with reference to any other moving particle is called a ***relative quantity.***

10. Relative Velocities. Consideration of the following examples will bring out the difference between absolute and relative quantities as defined.

EXAMPLE 1. Two sliding blocks, A and B, start from the same point and travel along a straight slot, both moving in the same direction. After an interval of time, A has an absolute velocity of 10 feet per minute, and B has an absolute velocity of 20 feet per minute. The velocity of B relative to A is 20 minus 10, or 10 feet per minute, the sense of this relative velocity being the same as the absolute velocity of B. The velocity of A relative to B is also 10 feet per minute, but the sense of this relative velocity is opposite to the absolute velocity of A.

EXAMPLE 2. Suppose the two blocks started as before, but moved in opposite directions, their absolute velocities being 10 and 20 feet per minute, as before. The velocity of B relative to A is now 20 plus 10, or 30 feet per minute, the sense of this relative velocity being the same as the absolute velocity of B. The velocity of A relative to B is also 30 feet per minute, but the sense of this relative velocity is the same as that of the absolute velocity of A, or opposite to that of B.

Example 3. Suppose that the two blocks started from the same point, but traveled at right angles to each other. The relative velocities cannot be obtained by simple addition or subtraction, but must be found by vector differences. In Fig. 8, let OA represent, to scale, the absolute velocity of A, 10 feet per minute, and OB the absolute velocity of B, 20 feet per minute. The velocity of B relative to A is the vector difference $(OB - OA) = AB$, or 22.36 feet per minute, in a direction parallel to AB and toward B. The velocity of A relative to B is the vector difference $(OA - OB) = BA$, or 22.36 feet per minute parallel to BA and toward A.

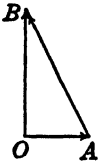

Fig. 8

Finding the relative displacements, velocities, or accelerations of two particles is the process of finding the vector difference of their absolute displacements, velocities, or accelerations. In every case the displacement (velocity or acceleration) of the particle A relative to the particle B is the absolute displacement (velocity or acceleration) of A minus the absolute displacement (velocity or acceleration) of B.

11. Mathematical Addition of Vector Quantities. In all of the preceding discussions, graphical solutions have been used. If extreme accuracy is desired, mathematical solutions may be employed, and in a few cases may be quicker and more readily understood than the graphical solutions.

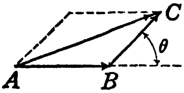

Fig. 9

Let AB and BC, in Fig. 9, represent two vector quantities. Combining these graphically gives AC as their resultant or sum. Let θ be the angle between the vectors, or the difference in their directions. From the trigonometric relations of the parts of a triangle it follows that

$$(15)\quad \text{vector sum } (AB + BC) = AC = \sqrt{AB^2 + BC^2 + 2 \times AB \times BC \times \cos\theta}.$$

Similarly, in Fig. 6, BC is the vector difference of AC and AB. Let ϕ be the angle between the vectors. Then

$$(16)\quad \text{vector difference } (AC - AB) = BC = \sqrt{AC^2 + AB^2 - 2AB \times AC \times \cos\phi}.$$

To find the sum of three or more vectors, it is convenient to break each vector into two components parallel to two reference axes and at right angles to each other. If X is the algebraic sum of all the components along one of these axes, and Y is the algebraic sum of all the components along the other axis, then

$$\text{vector sum} = \sqrt{X^2 + Y^2}. \tag{17}$$

The direction of the vector sum is determined by the relation

$$\tan \theta = \frac{Y}{X}, \tag{18}$$

where θ is the angle between the vector sum and X axis. The sense of the vector sum is determined by the sense of X and Y.

PROBLEMS

1. A force of 100 lbs. acting at an angle of 45° with the horizontal is to be added to a force of 150 lbs. acting vertically downward. Determine the magnitude and direction of the resultant.

2. Find the resultant of the following force system, assuming all forces applied at the same point on the body: forces of 2000, 2500, 4000, and 1500 lbs. acting at 30°, 0°, 120°, and 210° with the X axis.

3. An overhead crane moves forward at a velocity of 30 ft. per min., and the hoisting carriage moves across the crane at a velocity of 2 ft. per sec. What is the absolute velocity of the hoisting carriage?

4. A crane moves forward with a velocity of 25 ft. per min. while the hoisting block rises with a velocity of 5 ft. per min. What is the absolute velocity of the hoisting block?

5. A stream of water leaves a nozzle with a horizontal velocity of 600 ft. per min.
   (a) Find the absolute velocity of a particle of water 2 sec. after leaving the nozzle? State the magnitude and direction.
   (b) What will be the absolute acceleration of this particle at the same instant?

6. A small steel ball is subjected to the action of three magnetic forces which tend to give it accelerations of 1, 2, and 2½ ft. per sec. per sec. in directions 0°, 45°, and 60° with the horizontal. Determine the magnitude, direction, and sense of the resultant acceleration, graphically and mathematically.

7. A vector representing a velocity of 50 ft. per min. acting at 45° with the horizontal is to be broken up into two vectors representing velocities acting along the horizontal and at 120° with the horizontal. Determine graphically the magnitudes of the two component velocities.

8. A force of 100 pounds acting at 30° with the horizontal is to be replaced by a force acting at 60° and one acting at 330° with the horizontal. Determine the magnitudes of the replacing forces.

9. A beam weighing 500 lbs. has its ends resting on smooth planes making angles of 15° and 120° with the horizontal. Determine the pressure on each end of the beam due to the reactions perpendicular to the supporting planes.

10. Two forces are acting through a given point in a body. One has a magnitude of 100 lbs. and acts along the horizontal and to the right. The second has a magnitude of 75 lbs. and acts vertically up. Replace the two forces by a force of 50 lbs. acting along the horizontal and to the left, and two forces of unknown magnitude acting at 60° and 30° with the horizontal. Find the magnitudes of the unknown forces.

11. Using the scales indicated, subtract the following vector quantities. State numerical results and indicate the direction and sense of the difference.

(a) 40 ft. per sec. at 90° from 60 ft. per sec. at 30°. Scale 1 in. equals 50 ft. per sec.
(b) 75 ft. per sec. per sec. at 0° from 100 ft. per sec. per sec. at 120°. Scale 1 in. equals 200 ft. per sec. per sec.
(c) 20 lbs. at 135° from 10 lbs. at 30°. Scale 1 in. equals 10 lbs.
(d) 60 lbs. at 270° from 60 lbs. at 120°. Scale 1 in. equals 50 lbs.

12. An airplane is traveling due north at 100 mi. per hr. The wind is blowing from the northeast at a velocity of 30 mi. per hr. What is the velocity of the wind relative to the airplane?

13. Two automobiles collide at a street corner. The automobile A was traveling north at 20 mi. per hr., and the automobile B was traveling west at 30 mi. per hr.

(a) What was B's velocity relative to A when they collided?
(b) In what direction relative to B was A traveling at the instant of collision?

14. A lathe tool cuts threads on a steel bar so that there is one thread per in. The lathe spindle revolves at 60 r.p.m., giving the bar a surface velocity of 30 ft. per min. What is the velocity of the tool relative to the surface of the bar?

15. The blades of a steam turbine have a peripheral speed of 600 ft. per sec. The steam entering the blades has a velocity of 1200 ft. per sec. and is directed at an angle of 15° with the path of the blades. Determine the velocity of the steam relative to the blades.

16. The nut on a threaded bar moves axially along the bar ½ in. per revolution of the bar. If the bar turns at 25 r.p.m., with a surface speed of 10 ft. per min., what is the velocity of a point on the surface of the bar relative to the nut?

17. A 12-in. grinding wheel rotates with a surface speed of 2500 ft. per min. and is used to grind the surface of a 3-in. cylinder which revolves with a surface speed of 150 ft. per min.

(a) If the grinding wheel and the cylinder revolve in the same direction, what is the cutting speed?
(b) If the grinding wheel and the cylinder revolve in opposite directions, what is the cutting speed?

18. Water entering the blades of a water-wheel has a velocity of 40 ft. per sec. in a direction of 15° with the direction of motion of the blades. The blades are moving with a velocity of 22 ft. per sec. What is the velocity of the water relative to the blades? State the magnitude and direction.

19. A slide is moving outward in a radial groove cut in a rotating plate, at a rate of 3 in. per sec. The plate is revolving at a rate of 3 r.p.m. Determine the absolute velocity of the slide when it is 1 ft. from the center of rotation.

20. An airplane, A, has a velocity of 100 mi. per hr. directly south while a second airplane, B, has a velocity of 80 mi. per hr. northeast. Determine the velocity of B relative to A.

21. Using a scale of 1 in. equals 10 ft. per sec. velocity, 20 ft. per sec. per sec. acceleration, or 50 lbs. force, add the following quantities and give complete numerical values of the sums. Angles are measured counter-clockwise from a horizontal axis.

(a) 20 ft. per sec. at 0°, 15 ft. per sec. at 30°.
(b) 10 ft. per sec. at 30°, 12 ft. per sec. at 90°, 6 ft. per sec. at 135°.
(c) 15 ft. per sec. per sec. at 300°, 30 ft. per sec. per sec. at 180°, 19 ft. per sec. per sec. at 90°.
(d) 200 lbs. at 45°, 80 lbs. at 270°, 20 lbs. at 135°, 75 lbs. at 60°, 150 lbs. at 180°.

22. Using the scales indicated, resolve the vectors indicated into components acting in the directions indicated. State numerical results and indicate the sense of each component.

(a) 100 lbs. at 30° into components at 60° and 180°.
Scale 1 in. equals 40 lbs.
(b) 50 ft. per sec. at 90° into components at 210° and 45°.
Scale 1 in. equals 50 lbs.
(c) 250 lbs. at 135° into components at 90° and 210°.
Scale 1 in. equals 200 lbs.
(d) 75 ft. per sec. at 225° into components at 60° and 135°.
Scale 1 in. equals 50 ft. per sec.

CHAPTER III

MOTION OF A BODY IN SPACE

12. Particles. A body so small that its dimensions are negligible when compared to its range of motion, is called a ***particle.*** A particle may be considered as having the same velocity, acceleration, and displacement in all its parts.

A ***body*** is made up of a large number of particles, and the motion of the body may be studied by considering the velocities, accelerations, and displacements of its component particles.

13. Newton's Laws of Motion. The student has studied Newton's Laws of Motion in elementary physics; but since these laws are fundamental in the study of the motion of a body, they are included here.

(1) When a particle is not acted upon by any force, if at rest it remains at rest, and if in motion it continues to move in a straight line with uniform velocity.

(2) When a particle is acted upon by a single force, it is accelerated in the direction of the force. The magnitude of the acceleration is proportional to the force and inversely proportional to the mass of the particle:

$$a = \frac{F}{M}.$$

(3) When one particle exerts a force on a second particle, the second particle exerts an equal and opposite force on the first, or, as often stated, action and reaction are equal and opposite.

In this chapter and in the chapters immediately following, the chief considerations are the velocities of particles and the transmission of velocity from one particle to another. It should be realized, however, that whenever there is a change of velocity, either in magnitude or in direction, there must be a force action, and that the magnitude of this force is of considerable importance in the analysis.

14. Translation. When a rigid body moves in such a way that every straight line on the body remains fixed in its original direction, that is, parallel to its original position, the motion is said to be a ***translation.*** Translation does not require that every particle of the body move in a straight line, but requires that all particles at each instant have the same velocity, acceleration, and displacement. Hence the motion of any particle of a body in translation defines the motion of every other particle of the body.

An example of translation is found in the movement of the side rods of a locomotive when the locomotive runs on a straight track without rolling or nosing.

When every particle of the body moves in a straight line, there results a special case of translation called ***rectilinear translation.*** This is the type of translation ordinarily considered. The frame of the locomotive mentioned above would have rectilinear translation.

15. Rotation. When a rigid body moves in such a way that the position of one line in the body, or in an imaginary extension of the body, remains fixed, the motion is ***rotation.*** The fixed line is the axis of rotation. All particles of a rotating body move in circular paths about the axis of rotation.

16. Angular Velocity. By definition, angular velocity is the time rate at which a body turns or rotates about an axis of rotation. If one considers a plane cut through the body perpendicular to the axis of rotation, it is evident that all lines in this plane which pass through the axis of rotation have the same angular velocity. It is demonstrable that all lines in this plane have the same angular velocity.

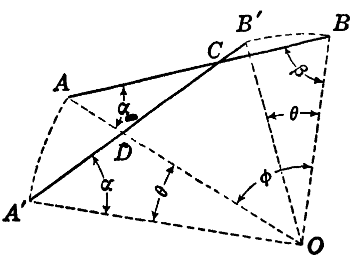

FIG. 10

In Fig. 10, let O be the intersection of the axis of rotation with a plane perpendicular to the axis. Let AB be any line in this plane. Rotate the body at a uniform rate through the angle θ

until AB takes the position $A'B'$. During this process θ/t is the angular velocity of OB and OA. From Fig. 10, we see that

$$\angle ADC = \angle A'DO = 180 - (\alpha + \theta),$$
$$\angle B'CB = \angle ACD = 180 - (\angle ADC + \alpha);$$

therefore

$$\angle B'CB = 180 - [180 - (\alpha + \theta) + \alpha] = \theta.$$

It follows that the angular displacement of AB is equal to the angular displacement of OB, OA, or any other line in the plane. Since angular displacement divided by time is angular velocity, we see that at each instant all lines in the plane perpendicular to the axis of rotation have the same angular velocity.

Angular velocity is generally stated in radians per second, or in revolutions per minute. Since one revolution equals 2π radians, it follows that

$$\omega = \frac{2\pi N}{60},$$

where

ω = angular velocity in radians per second,
N = revolutions per minute.

17. Linear Speed of a Particle on a Rotating Body. All particles on a rotating body have the same instantaneous angular velocities, but the linear speeds vary as the distances of the particles from the axis of rotation. The linear speed of any particle of a body which is rotating with uniform angular velocity is constant in magnitude but constantly varying in direction. If a particle constrained to move in a curved path suddenly has the constraint removed, that is, has no force acting on it, the particle will, according to Newton's First Law, continue to move in a straight line tangent to the path at the point where the constraint is removed. At any instant, then, every particle in the body is moving perpendicular to the line joining that particle and the center of rotation.

The linear speeds of the particles are proportional to the length of the lines joining the particles and the center of rotation. In Fig. 10, when the body revolves through an angle θ, the particle

A traverses the arc AA', and the particle B traverses the arc BB'. We have

$$\text{linear speed of } A = \frac{\text{arc } AA'}{\Delta t} = \frac{\theta \times OA}{\cdot\Delta t} = \omega r_a = \frac{2\pi r_a N}{60},$$

$$\text{linear speed of } B = \frac{\text{arc } BB'}{\Delta t} = \frac{\theta \times OB}{\Delta t} = \omega r_b = \frac{2\pi r_b N}{60},$$

$$(19)\quad \frac{\text{linear speed of } A}{\text{linear speed of } B} = \frac{r_a}{r_b},$$

and also

$$(20)\quad \text{linear speed of } A - \text{linear speed of } B = \omega(r_a - r_b).$$

18. Plane Motion in General. When every particle of a body remains at a fixed distance from a fixed plane, that is, when all particles move in parallel planes, the body is said to have ***plane motion.***

In plane motion, all particles which lie on a laterally moving line which is perpendicular to the fixed plane, or plane of motion, have the same motion as the point projection of the perpendicular line on the plane of motion. One can therefore study the motion of the body by considering only the point projections on the plane of motion.

Plane motion of a body can be accomplished by:

(*a*) simple translation.
(*b*) simple rotation.
(*c*) translation followed by rotation.
(*d*) rotation followed by translation.
(*e*) combined rotation and translation.

In Fig. 11, the body ABC * moves from the position ABC to the position $A'B'C'$. This motion can be accomplished by translation followed by rotation. First, slide the body to the position $A'B''C''$, keeping $A'B''$ parallel to AB, and $B''C''$ parallel to BC, that is, keeping each line in the body always parallel to its original position; and second, rotate the body about the point A' to the position $A'B'C'$ in the same plane.

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* The motion of a body in a plane is determined if the motion of two points on the body is defined; therefore the examples represent general cases and the points of the triangles are points on any body having plane motion.

The same total displacement can be obtained by first rotating the body about the point A, to the position $AB'''C'''$ until AB''' is parallel to $A'B'$, and second, translating the body with each line in the body always parallel to its second position, to the position $A'B'C'$.

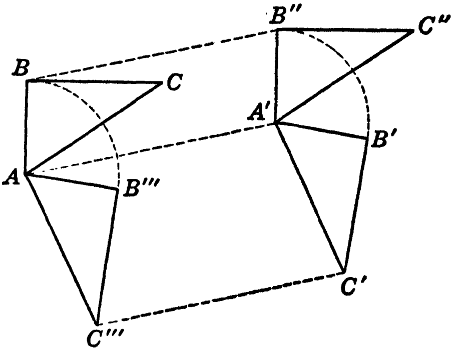


Fig. 11

The same total displacement may also be accomplished by a single rotation. In Fig. 12, the body ABC has moved over some path to the position $A'B'C'$. The point A can be moved to A' by rotation about any point on the line DO, the perpendicular bisector of AA'. Also C can be moved to C' by rotation about any point on EO, the perpendicular bisector of CC'. Since the point O is common to both of these bisectors, the body can be rotated by a single rotation about an axis at O to the position $A'B'C'$.

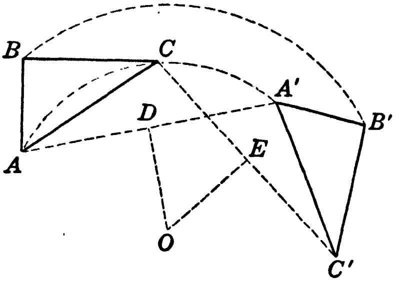


Fig. 12

19. Instantaneous Plane Motion. Both the translation and rotation of the body may take place simultaneously; and if an infinitesimal time interval is considered, one can imagine the infinitesimal displacement of the body as made up of an infinitesimal translation and an infinitesimal rotation. Also, since the linear velocity is the displacement in a straight line, divided by the time interval, and the angular velocity is the angle of rotation divided by the time interval, one can imagine any ***instantaneous velocity*** of a body as made up of a linear velocity and an angular velocity about some axis.

For example, in Fig. 13, a ladder AB rests on the floor at A and on the wall at B. When A is moved along the floor to a point A', B will move down the wall to B'. The total displacement of the ladder may be accomplished if the ladder is translated to $A'B''$, and then rotated to $A'B'$. If these motions occur simultaneously, the instantaneous velocity of B is made up (1) of a velocity equal and parallel to that of A, and (2) of a velocity due to the rotation of AB about A. The combination of these velocities is such that B has an actual velocity downward along the wall. Similarly, the actual velocity of a point C is a velocity equal and parallel to the velocity of A, combined with a velocity perpendicular to AC and due to rotation about A.

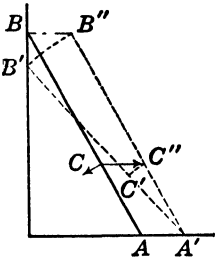


FIG. 13

20. Instantaneous Center of Rotation. In § 18, it is shown that any plane motion of a body may be accomplished by a single rotation about some center of rotation. In actual continuous motion from position ABC to position $A'B'C'$, Fig. 14, the particles of the body may not have followed circular arcs about a single point, but may have followed some irregular paths. The actual paths of the particles may be closely approximated by a series of rotations about centers O_1, O_2, etc., into intermediate positions $A_1B_1C_1$, $A_2B_2C_2$, etc., until the final position $A'B'C'$ is reached. The smaller these successive rotations become, the more nearly the actual motion of the body is duplicated. With infinitesimal rotations the actual motion is exactly duplicated, the centers of rotation forming a continuous line.

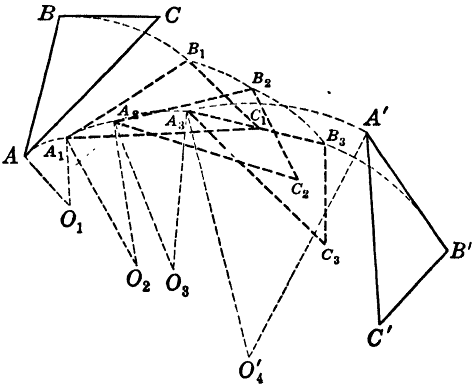


FIG. 14

The motion of a body may thus be duplicated by a continuous rotation about a center of rotation which is itself moving. The position of the point about which the body is rotating at any instant is called the ***instantaneous center of rotation,*** or, for the sake of brevity, ***instantaneous center.***

The actual motion of a particle in the body at any instant can be considered as a rotation about an instantaneous center on a line perpendicular to the path. There may be an infinite number of these centers, but each will be on the line perpendicular to the tangent to the path of the particle at the instant under consideration. If the direction of the motion of a second particle in the body is known, then the instantaneous center for the body is located at the intersection of the lines drawn through the two points perpendicular to their respective motions. The instantaneous plane motion of all particles of the body is then completely determined.

PROBLEMS

1. The piston of an internal-combustion engine has a maximum acceleration of 1000 ft. per sec. per sec. What is the force required to produce this acceleration if the piston weighs 2 lbs.?

2. An electric street-car weighing 35,000 lbs. starts from rest and attains a speed of 25 mi. per hr. in ⅛ min.
   - (a) What is the average acceleration in ft. per sec. per sec.?
   - (b) What is the average acceleration in mi. per hr. per min.?
   - (c) What force must be exerted by friction between the driving wheels and the rails to produce this acceleration?

3. A freight car weighs 40,000 lbs. and carries a load of 30,000 lbs. It is accelerated at the rate of 180 ft. per min. per min.
   - (a) What force at the draw-bar is required to produce this acceleration?
   - (b) How many mi. per hr. will the speed of the car be increased in 5 min.?

4. The cutting speed of a certain tool used on steel is 100 ft. per min. At what rate should the lathe revolve when machining the outside of a bar 8 in. in diameter?

5. The outside diameter of the rotor of a steam turbine is 3 ft. 2½ in., and the linear speed of a point on the circumference is 10,000 ft. per min. Determine the angular velocity of the turbine in r.p.m. and in radians per sec.

6. Examine the equipment in the laboratory and the machine-shop of the Mechanical Engineering Department, and name at least 10 uses of parts having motion of translation. Which of these have rectilinear translation?

7. Examine the equipment in the laboratory and machine-shop, and name at least 10 uses of rotating parts.

8. How many r.p.m. are made by a locomotive driving-wheel that is 6 ft. in diameter if the train is running 40 mi. per hr.?

9. An automobile tire is 32 in. in diameter and the automobile is running 30 mi. per hr.

(a) What is the speed of the tire in r.p.m.?
(b) What is the angular velocity in radians per sec.?

10. An internal-combustion engine has a crank 2½ in. long and runs at 1000 r.p.m.

(a) What is the linear speed of the crank-pin in ft. per min.?
(b) What is the average velocity of the piston in ft. per min.?

11. The linear speed of the rim of the rotor of a steam turbine is 25,120 ft. per min.

(a) What is the speed of rotation in radians per min. if the diameter is 48 in.?
(b) What is the speed of rotation in r.p.m.?

12. The crank-pin of a steam engine has a linear speed of 541 ft. per min. The length of the crank arm is 9 in. Attached to the same shaft is a fly-wheel 8 ft. in diameter.

(a) How many revolutions per min. is the engine making?
(b) What is the linear speed of the outer surface of the wheel?

13. A gear 5 ft. in diameter is running 200 r.p.m.

(a) What is the speed of a point on its outer surface?
(b) What is the angular velocity in radians per sec.?

14. A point on a fly-wheel has a linear speed of 1500 ft. per min. less than the speed of a point on the rim. If the fly-wheel has an angular velocity of 5 radians per sec., what is the radial distance between the two points?

15. A wheel 3 ft. in diameter rotates at 300 r.p.m. Two points, *A* and *B*, lie on radii making an angle of 60° with each other. The points are ½ ft. and 1 ft. respectively from the center of rotation.

(a) What is the linear speed of *A*?
(b) What is the angular speed of the line *AB*?

16. A vertical line *AB* is 2 in. long. This line moves in such a way that *AB* comes to rest on a 45° line with point *A* 1 in. directly below its original position. Locate graphically the center about which *AB* could have been rotated in order to accomplish this displacement by a single rotation.

17. Two particles on a link are 4 in. apart. One particle is moving on a horizontal line and the other particle is 1 in. above this horizontal and is moving in a direction making an angle of 60° with the horizontal. Locate the instantaneous center of the link.

18. A ladder 10 ft. long is resting on the floor and against a vertical wall. If the lower end is 3 ft. from the wall and is being pushed toward it, locate the instantaneous center of rotation.

19. Two points on a rotating disk have a difference in linear speed of 6000 ft. per min. The difference in their radii is $2\frac{1}{2}$ ft.

(a) What is the angular velocity of the disk in radians per sec.?

(b) What is the angular velocity in revolutions per min.?

20. A crank rotating at 45 radians per sec. carries two pins, *A* and *B*. The pin *B* is to have a linear velocity 180 ft. per sec. greater than the linear velocity of *A*. Determine the difference in the distances of *A* and *B* from the center of rotation.

21. One end of the connecting-rod of a steam engine is moving to the right on a horizontal line with a velocity of 500 ft. per min. The other end is moving up at an angle of 45° with the horizontal. The rod is 4 ft. long and makes an angle of 15° with the horizontal. Locate the instantaneous center of the connecting rod.

22. One end of a 6-ft. rod is resting on a plane making an angle of 45° with the horizontal. The other end is supported by a rope making an angle of 75° with the horizontal. The rod itself makes an angle of 15° with the horizontal. If the rod end is sliding down the plane locate the instantaneous center of the rod.

23. A locomotive is traveling at the rate of 35 mi. per hr. and has driving wheels 63 in. in diameter and steam cylinders with 24-in. stroke.

(a) Find the angular velocity of the crank pin.

(b) Find the linear velocity of the crank pin relative to the engine frame.

(c) Find the linear velocity of the crank pin relative to the track when the crank pin is directly below the axle.

CHAPTER IV

THE FOUR-BAR LINKAGE

21. Linkwork. A linkwork or linkage, as applied to mechanisms, means a combination of a number of pairs of elements, such as levers, cranks, slides, etc., connected by rigid pieces or links, all the parts being connected by pin or pivoted joints allowing relative motion between the parts. All the parts must be so connected that when any one part is moved, definite motion is imparted to all the other parts, except when the links are in the position known as the dead point as explained in § 30.

22. Four-Bar Linkage. Four rigid pieces pinned together so that there can be definite relative motion between the parts is called a ***four-bar linkage.*** In order that any linkage may transmit definite relative motions, the relation between the pin joints and the rigid pieces must be such that the linkage forms a four-bar linkage, or a combination of four-bar linkages.

In Fig. 15, three rigid pieces or links are pinned together at *a, b,* and *c* to form a triangle. When one of these links is moved, the others must move also, keeping the form of the triangle the same. There is no relative motion between the links themselves. This combination is not a linkage, but is simply the equivalent of a rigid body.

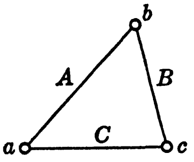


Fig. 15

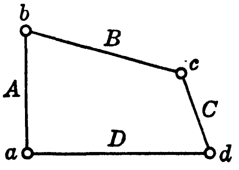


Fig. 16

In Fig. 16, four links are pinned together at *a, b, c,* and *d.* This linkage is possible; and, when one link is held and any other link is moved, definite relative motion is imparted to the remaining two links.

In Fig. 17, five links are pinned together at *a, b, c, d,* and *e.* In this case, relative motion of the links is possible, but it is evident that the relative motions of the links are not determined, since some of the links may be moved without transmitting definite motions to the other links. This is not a possible linkage in any mechanism.

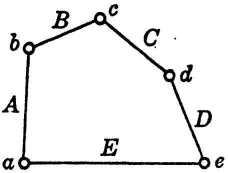


FIG. 17

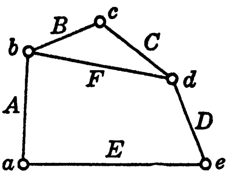


FIG. 18

In Fig. 18, an additional link has been added to the system used in Fig. 17. Relative motion of the links is now fully determined, the three links forming the triangle *bcd* being rigid and equivalent to a single link *F*. This system is, therefore, a four-bar linkage.

Figure 19 shows a linkage in which definite relative motions of the links is possible, and which can be broken up into two four-bar linkages *ABFE* and *FCDE*.

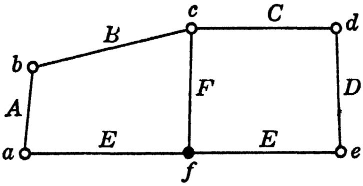


FIG. 19

If the frame, or fixed link, of a four-bar linkage is itself a moving link of a second linkage, the motions of the links of the first linkage, relative to each other, remain the same as if the frame were stationary. Their absolute velocities, however, are different from what they would be if the frame were stationary.

23. Path of a Point in a Linkage. If a particle is not connected to other particles, or if, when connected, it is not forced to travel in any definite path, it is called a free particle and has free motion. If the particle is a part of a linkage, or part of a body connected to other bodies having definite relative motions, it is a constrained particle and has constrained motion. All linkages, mechanisms, and machines depend upon constrained motion for their proper action.

In the simple four-bar linkage of Fig. 20, consider that the crank *A* revolves about the center *a* which is fixed on a frame *D*. Then all particles of the crank, such as *b* and *p*, move in circular paths about *a*. All points in the crank *C*, such as *c* and *m*, oscillate in circular arcs about the center *d* which is also fixed in the frame *D*. Their paths are not complete circles, since the crank *C* is constrained to move between the positions C_1 and C_2.

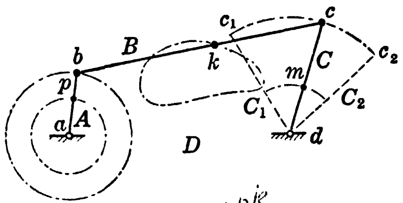


FIG. 20

[illegible] link

On the link *B*, the particles move in various paths, their motion being determined by the rotation of the crank *A* and the oscillation of the crank *C*. The path of a point *k* on the link *B* is indicated.

24. Sliding Members. Figure 21 shows a four-bar linkage, *ABCD*, with the crank *C* replaced by a circular slot and sliding block *E*. If the circular slot has its center at *d* and a radius equal to the original crank *C*, then the point *c* will move in the same path when guided by the slot or the crank. Although there are apparently only three links, the crank *A*, the link *B*, and the frame *D*, this is a four-bar linkage, the slot and slide *E* being the fourth link and equivalent to the crank *C*.

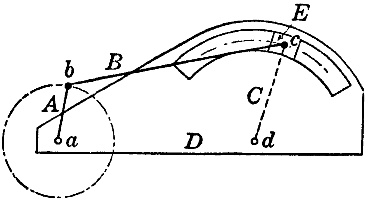


FIG. 21

If the radius of the slot, or the equivalent crank *C*, is made of infinite length, the slot becomes straight as in Fig. 22. This mechanism is found on the ordinary steam engine, the sliding block being the engine crosshead, and the straight slot being the crosshead guide. This mechanism is also a four-bar linkage.

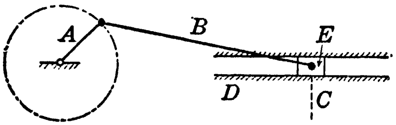


FIG. 22

Slider Crank

25. Relative Motions in Four-Bar Linkages. The four-bar linkage can be used to transmit a variety of relative motions from the driver to the follower by changing the proportions of the cranks and connecting links. The most important of these are illustrated in Figs. 23 to 26. In these figures and in the descriptions which follow, the crank A is the driver, the crank C is the follower, and the link D is the frame or fixed link.

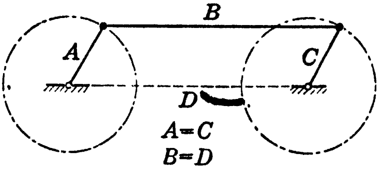


FIG. 23

26. Parallel Links. In Fig. 23, two *equal cranks are connected by a link equal in length to the fixed link.* When A revolves, C revolves in the same direction and with the same angular velocity, B being always parallel to D.

27. Crossed Links. In Fig. 24, two *equal cranks are connected by a link equal in length to the fixed link but crossing it.* When A revolves, C revolves in the opposite direction. If the angular velocity of A is constant, the angular velocity of C always varies.

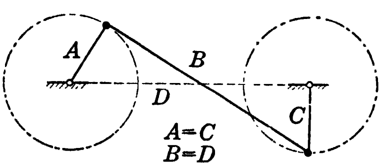


FIG. 24

28. Unequal Cranks. In Fig. 25, two *unequal cranks are connected by a link which is longer than the difference of the crank lengths plus the length of the fixed link, but shorter than the sum of the crank lengths minus the length of the fixed link.* When A revolves, C revolves in the same direction, both making complete revolutions. If the angular velocity of A is constant, the angular velocity of C always varies. This mechanism is known as the drag-link mechanism.

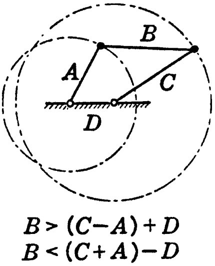
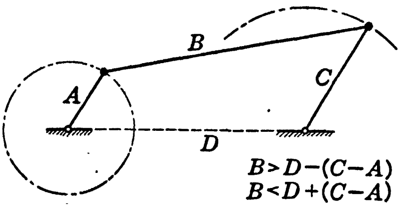


FIG. 25

29. Oscillating Links. In Fig. 26, *two cranks of unequal length are connected by a link which is longer than the fixed link minus the difference of the two cranks, but shorter than the fixed link plus the difference of the two cranks.* When A revolves, C oscillates. If the angular velocity of A is constant, the angular velocity of C always varies, and the time required for the forward stroke is different from the time required for the return stroke.



$B > D-(C-A)$
$B < D+(C-A)$

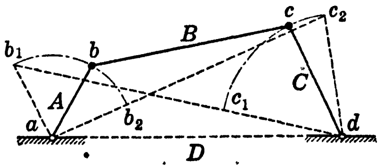
FIG. 26

By varying the relative lengths of the cranks A and C and the links B and D, the student can develop a number of interesting cases in which A and C oscillate. Both may oscillate on one side of the link D, or either one or both may be made to oscillate on both sides of D.

30. Dead Points. The linkages just discussed may not always give definite relative motion. Examination of the linkage in Fig. 23 shows that when the connecting link B and the cranks A and C all lie in the same straight line, a rotation of A downward may cause the crank C either to rotate downward, maintaining a parallelogram, or to rotate upward, making the connecting link cross the fixed link, in which case the linkage becomes that of Fig. 24.

The same condition is found in several other linkages. Whenever the connecting link and one of the cranks lie in the same straight line, there is a possibility of two distinct motions. When a linkage is in such a position that definite motion cannot be transmitted to the follower or driven link, it is said to be on a *dead point.*

The dead point of a simple linkage such as that shown in Fig. 27 can be located by an arc with d as a center and a radius equal to C plus B, intersecting the circle of radius ab at b_1. This is a dead point, since motion of the driving crank A from the position b_1 may cause the crank C to move in either direction from the position c_1. Note that some



$B < D-(C-A)$

FIG. 27

means of starting the crank C in the direction desired must be provided, since a force applied along B will not move C when B and C are in a straight line, that is, when C is on a dead point.

In like manner, if the crank C is the driver, the linkage will be on a dead point when A and B are in a straight line ab_2c_2. Dead points must always be provided against in mechanisms by means of auxiliary links, springs, or inertia.

31. Relative Motions of Linkages with Sliding Members. If a sliding member is substituted for the crank C, Fig. 20, four important combinations occur, in which A, B, C, and D in turn become the fixed links. These four combinations form the basic linkages of many groups of useful machines.

32. First Group. In Fig. 28, the linkage consists of a crank A, a link or connecting rod B, a slide or crosshead C, and a slot or crosshead guide in the link D. The link D is fixed, and the connecting rod is longer than the crank or $B > A$.

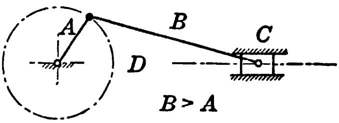


FIG. 28

When A rotates as a driver, the crosshead C reciprocates in the slot. This linkage is called the crank-and-pitman linkage.

When the crosshead C reciprocates as the driver, the crank A rotates. This is the crosshead-and-crank linkage, often called the steam-engine mechanism.

33. Second Group. In Fig. 29, the link A, or crank, of the crank-and-pitman linkage is now fixed and shorter than B. When B is the driver and rotates with a uniform angular velocity, D also rotates, but with varying angular velocity. This linkage is used in the Whitworth quick-return motion.*

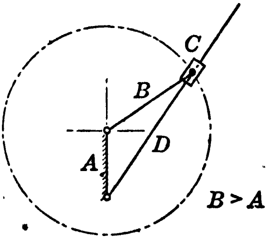


FIG. 29

When B rotates with uniform velocity, D rotates with varying angular velocity.

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* This linkage may be found on shapers or similar metal cutting machines where a quick-return stroke is desired. See §§ 118 and 119.

The linkage is the same as the driving mechanism of the Gnome and other rotating internal-combustion engines, where the sliding block *C* is the piston, and the link *D* is expanded to form a cylinder in which *C* slides.

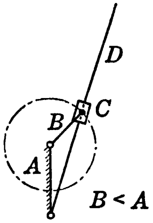


FIG. 30

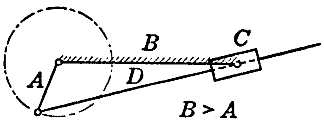


FIG. 31

When *B* is shorter than *A*, as in Fig. 30, and *B* rotates at uniform angular velocity, *D* will oscillate. This linkage is used in ~~the rocking-arm~~ quick-return motion, and may be found on the same type of machines as the Whitworth quick-return motion referred to above.

34. Third Group. In Fig. 31, the link *B* is fixed and longer than *A*. When *A* is rotated at uniform angular velocity, *D* slides in the crosshead *C* and oscillates. This linkage forms the mechanism of the oscillating steam engine.

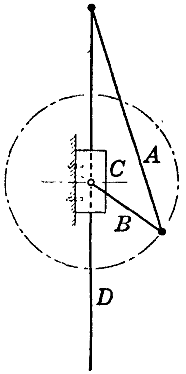


FIG. 32

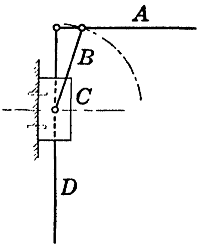


FIG. 33

35. Fourth Group. In Fig. 32, the slide, or crosshead, is fixed. When *B* rotates, the link *D* reciprocates. This form of the linkage is seldom used except on some types of windmills.

When the link *A* is extended, Fig. 33, and used as the driver with an oscillating motion, the linkage of the common house pump is obtained. This is also a form of linkage used on some small power-driven well pumps and air-tank pumps.

36. Compound Linkage. The linkage shown in Fig. 19 consists of six links which can be considered to constitute two four-bar linkages, namely *ABFE* and *FCDE*. In § 24, it was shown that a sliding member with its guide was equivalent to an imaginary crank, and that Fig. 21 and Fig. 22 were equivalent to four-bar linkages. Every common mechanism consisting of combinations of four or more links or sliding members can be broken up into one or more simple four-bar linkages or their equivalents. The preceding discussions, therefore, apply to all linkages or linkwork.

PROBLEMS

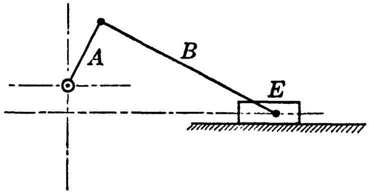
1. Separate the linkage of Fig. 40 into its simple four-bar linkages, indicating the imaginary equivalent cranks.

2. Separate the linkage of Fig. 41 into its simple four-bar linkages, indicating the imaginary equivalent cranks.

3. Separate the linkage of Fig. 77 into its simple four-bar linkages, indicating the imaginary equivalent links.

4. Separate the linkage of Fig. 106 into its simple four-bar linkages, indicating the imaginary links.

5. In the figure let the slide *E* be the driver, and make the center-line of its path pass ½ in. below the center of rotation of *A*. Make *A* = 1½ in., *B* = 4 in. Locate the dead-point positions.



PROBLEM 5

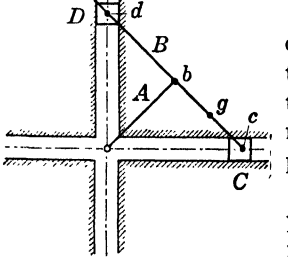
6. Two cranks, *A* and *C*, of a four-bar linkage have their centers 4 in. apart. Crank *A* is 2 in. long, and crank *C* is 3 in. long. The connecting-rod is 5 in. long.

(a) Determine the dead-point positions.

(b) Through what angle can the crank *C* move?

7. In the shops and laboratories of your department locate specimens of each of the types of linkages mentioned in Articles 26 to 36. State the name and location of the machine on which it is located, and state the purpose for which the linkage is used.

8. In the figure for Problem 5, trace the path of the midpoint of the link *B* when *A* makes a complete revolution.

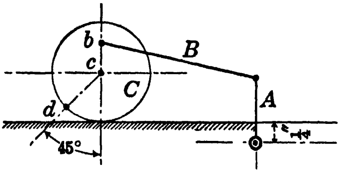


PROBLEM 9

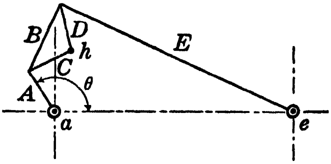
9. In the figure, make *A* = 1 in., *bc*

= bd = 1 in., dh = ¾ in., and gc = ½ in. Trace the paths of g and h when A makes a complete revolution.

10. In the figure make A = 1 in., B = 2½ in., C = 1½ in. diameter, and bc = ½ in. Plot the path of the point d and the mid-point of B when A makes a complete revolution.

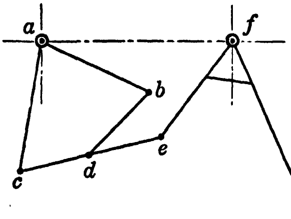


PROBLEM 10



PROBLEM 11

11. In the figure make A = 1 in., B = ¾ in., C = D = ½ in., E = 2¾ in., and ae = 2½ in. Plot the path of h when A makes a complete revolution.

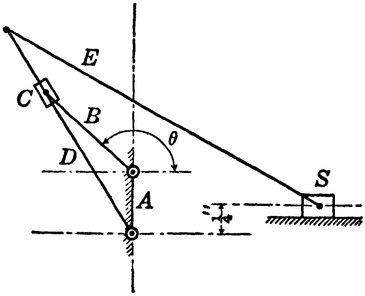


PROBLEM 12

12. In the figure, ab and ac turn freely on the fixed center a. Make af = 2 in., ab = 1¼ in., ac = 1⅜ in., ce = 1½ in., cd = ¾ in., bd = ⅞ in., ef = 1½ in., and fh = 2 in. The angle cfh is 60°. Plot the path of the point d when fh oscillates 30° on each side of the vertical line through f.

13. A crank A, 1 in. long, and a crank C, 1½ in. long, are connected by a rod 2 in. long, and the distance between fixed centers is 1¾ in. Determine the angular movements of the cranks A and C.

14. In the figure, make A = ½ in., B = 1 in., D = 2 in., and E = 3 in. Determine the positions of B when the slide S is at its end positions. Determine the angle traversed by the crank B when S is moved to the left and when moved to the right. B is the driving crank.



PROBLEM 14

15. Lay out a four-bar linkage which will allow both levers to oscillate on only one side of the line joining the fixed centers but will not allow either to cross this line.

16. A driving crank 1¼ in. long is to revolve at a uniform angular velocity. An oscillating lever whose fixed center is 2 in. from the

center of rotation of the revolving crank is to move through an angle of 45° on each side of a line perpendicular to the line of centers.

(a) Determine the length of the oscillating lever and of the connecting rod.

(b) Through how many degrees will the driving crank turn while the oscillating lever is moving toward the driver?

17. In the figure for Problem 14, make the length of the driving crank *B* equal to 3/8 in., the other dimensions remaining the same. Through what angle will *D* oscillate, and what will be the movement of *S*? If the link *E* is changed to 4 in., what will be the movement of *S*? If *D* is now changed to 2½ in., what will be the movement of *S*? Does changing the length of *E* or *D* affect the angular movement of *B*?

18. Using the figure of Problem 5, plot a diagram showing the movement of the slide *E* for equal angular movements of the crank *A*. Begin with *E* at its extreme position to the right. Plot the movement of *E* as ordinates, and the angular movement of *A* as abscissae to the scale 1 in. equals 120°.

19. Using the figure of Problem 10, plot a diagram showing the movement of the wheel center *c* for equal angular movements of the crank *A*. Begin with *A* in the vertical position. Plot the movement of *c* as ordinates and the angular movement of *A* as abscissae to the scale 1 in. equals 120°.

20. Using the figure for Problem 14, plot a diagram showing the movement of the slide *S* for equal angular movements of the crank *B*. Begin with the slide *S* at the extreme right-hand position. Plot the movement of *S* as ordinates and the angular movement of *B* as abscissae to the scale 1 in. equals 60°.

CHAPTER V

VELOCITIES IN LINKWORK

37. Use of Velocity Analysis. In many machines, it is desirable to study the changes in the velocities of certain parts during a complete cycle of motions of the linkages making up the machine. Velocity diagrams made from these studies are used for different purposes, depending on the use to which the machine is to be put. In the case of machine tools,* the velocity diagrams will show how nearly uniform the speed of the tool remains during the cutting stroke, as well as the ratio of the time required for the cutting stroke and for the return stroke in tools utilizing a reciprocating motion. In machines like steam engines and gas engines, especially those running at high speeds, the velocity diagrams may be used to determine accelerations of the parts, the accelerations in turn being used to determine the accelerating and inertia forces induced by the motion of the body. These forces should be known when the machine is designed, in order that the members may be made sufficiently strong, and that the machine may be properly balanced to reduce vibration.

In most machines the velocity of some point is known at all times during the complete cycle. From this known velocity, the velocity of any other point in the machine can be found for each instant during the cycle, and a velocity diagram may be plotted. The velocities can be found either graphically or mathematically or by a combination of both methods. Several methods of graphical solution will be developed, since the graphical are generally the most convenient methods to apply, although subject to the limitations of accuracy characteristic of drawing-board methods.

38. Velocity of Particles of a Rigid Body. In § 19, it is shown that the instantaneous displacement of a body can be imagined as the displacement of any particle, A, in the body combined with the rotation of the body about an axis through this particle, A. Stated in terms of vectors which represent the displacements, the displacement of particle B in the body can be found from the displacement of particle A by the relation:

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* A machine tool is a machine made for cutting or forming metals or woods by the use of a tool. Examples are lathes, planers, drill presses, and similar machines.

$$\Delta s_B = \text{vector sum } (\Delta s_A + \Delta s_{BA}), \tag{21}$$

where

Δs_B = displacement of B,
Δs_A = displacement of A,
Δs_{BA} = displacement of B due to rotation about A, or relative to A.

As these displacements are simultaneous, we can divide by the time interval and obtain the relation between the average velocities during the time interval Δt.

$$\frac{\Delta s_B}{\Delta t} = \text{vector sum } \left(\frac{\Delta s_A}{\Delta t} + \frac{\Delta s_{BA}}{\Delta t} \right), \tag{22}$$

or

$$\text{average } v_B = \text{vector sum (average } v_A + \text{average } v_{BA}), \tag{23}$$

during interval Δt.

When the time interval approaches zero, the average velocity becomes the ***instantaneous velocity*** of the particle.

$$v_B = \text{vector sum } (v_A + v_{BA}) = \text{vector sum } (v_A + \omega r_B), \tag{24}$$

where

v_B = instantaneous velocity of B,
v_A = instantaneous velocity of A,
v_{BA} = instantaneous velocity of B relative to A,
ω = angular velocity of B about A in radians per second,
r_B = distance from B to A.

In Fig. 34, let the particle A have an absolute velocity represented by the vector v_A, and let the body be rotating about A with an angular velocity ω. The velocity of B due to rotation about A, that is, relative to A, is $\omega \times AB$ and is represented by the vector v_{BA}. The absolute velocity of B is v_B, the vector sum of v_{BA} and v_A, mb in the figure being equal and parallel to Aa. Similarly, the velocity of C is the vector sum of v_A, the velocity of A, and v_{CA}, the velocity of C relative to A.

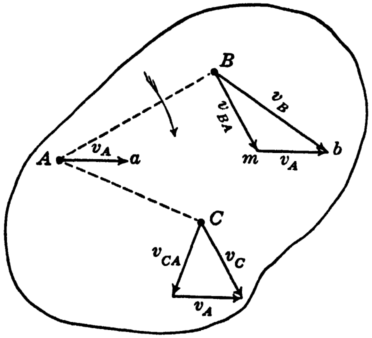


FIG. 34

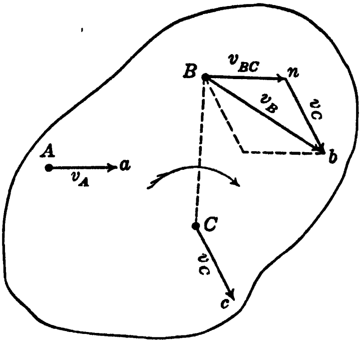


FIG. 35

In Fig. 35, which represents the same body as Fig. 34, assume that the velocity of C is the known velocity, and that its magnitude and direction are the same as those in Fig. 34. Now the velocity of B is the vector sum of v_C, the absolute velocity of C, and v_{BC}, the velocity of B relative to C, or $\omega \times CB$. The velocity of B is the same as found in Fig. 34.

That is, *the instantaneous velocity of any particle in a rigid body is the vector sum of the absolute velocity of any other particle in the body (or body extended) and the velocity of the first particle relative to the second.*

Also, if the absolute velocity of any particle of one of the bodies forming a mechanism is known, together with the angular velocity of the body, the absolute velocities of all particles in the body are completely determined and can be found by vector addition.

39. Velocity Transfer. In many cases, however, the angular velocity of the particular body is not known, but the direction of the velocity of a second particle in the body is known. In Fig. 36, assume that the velocity of A is known and represented by the vector Aa, and that the direction of the velocity of B is along the line Bd.

If *ac* and *bf* are drawn perpendicular to *AB*, it is readily seen that *Ac* and *Bf* are equal, since the component velocities of two particles in a rigid body along the line joining the particles must be equal. If this were not true, the particles would move farther apart or closer together, and the body would stretch or compress, a condition impossible in a rigid body.

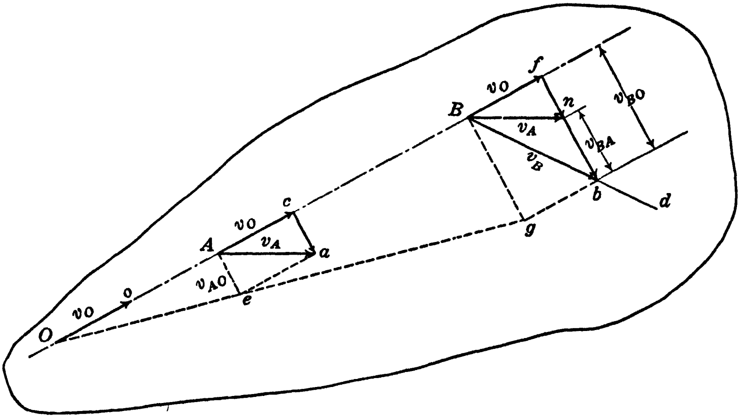


FIG. 36

The velocity of *B* relative to *A* is perpendicular to the line *AB;* and if *Bn* is laid off equal and parallel to v_A, and *nb* is drawn perpendicular to *AB*, the triangle *Bnb* will be a velocity polygon for *B*. The vector *Bb* represents v_B, the absolute velocity of *B*, and the vector *nb* represents v_{BA}, the velocity of *B* relative to *A*.

Draw *Bg* equal to *fb* and perpendicular to *AB*. Draw *Ae* equal to *ca* and perpendicular to *AB*. Draw *ge*, and extend to intersect *AB* extended at *O*.

A study of the figure also shows that

$$\frac{fb}{ca}=\frac{Bg}{Ae}=\frac{OB}{OA}=\frac{v_{BO}}{v_{AO}},$$

and that $Oo = v_0 = Ac$.

Therefore the absolute velocity of either *A* or *B* is seen to be the resultant of a velocity due to rotation about *O* and a velocity due to movement of *O* in the direction *AB*.

Applications of the foregoing principles to linkages will be developed by a number of illustrative problems.

40. Four-Bar Linkage. In the linkage of Fig. 37, the crank A is rotating about a, and at any instant the absolute velocity of b is perpendicular to ab. In the position shown, b has a velocity v_b represented to scale by the vector bb'. The vector components of bb' perpendicular and parallel to bc are bf and fb'. The absolute velocity of c must be perpendicular to the crank C, and the component of this velocity, along cb, must be equal to bf. The actual velocity of c is the resultant of the component cg and a component ch perpendicular to cb; the resultant cc' represents the velocity of c to the same scale that bb' represents the velocity of b.

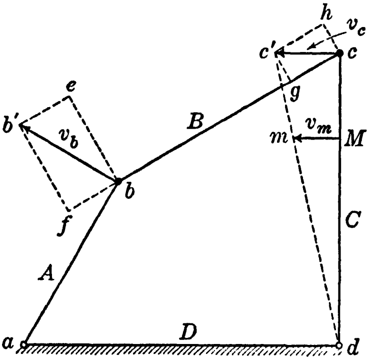


FIG. 37

The velocity of any particle on the crank C is perpendicular to dc and equal to the velocity of c times the ratio of the distance of these two particles from d. To find, graphically, the velocity of any particle, M, on the crank C, draw dc' and a line Mm through M, and perpendicular to dc, intersecting dc' at m.

$$v_M = \text{velocity of } M = v_c \times \frac{dM}{dc}.$$

Therefore Mm represents the velocity of M to the same scale that bb' represents the velocity of b, and that cc' represents the velocity of c.

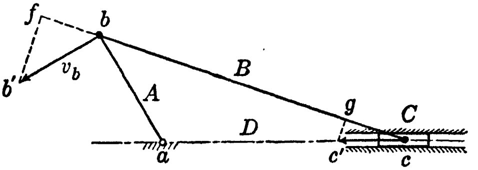


FIG. 38

41. Crank and Crosshead. In Fig. 38, the crank A is rotating, and its velocity, v_b, is represented to scale by the vector bb'. Resolve bb' into components bf and fb' parallel and perpendicular to *bc*. Draw *cg* equal to *bf*. The particle *c* is constrained by the guides to move in the direction *cc'*. Draw *gc'* perpendicular to *bc* and intersecting *cc'* at *c'*. The vector *cc'* represents the velocity of *c* to the same scale that *bb'* represents the velocity of *b*.

42. Sliding Members with Curved Guides. The method of velocity transference is the same as that given in the preceding article. The direction of the velocity of the sliding member is at any instant perpendicular to the radius of curvature of the path, that is, tangent to the path.

43. Particles Not at the Connecting Pins. In the four-bar linkage shown in Fig. 39, the particle *K* is on the link *B*, and on the straight line joining *b* and *c*. The particle *Q* is also on the link *B*, but is not on the straight line joining *b* and *c*. The crank *A* drives, and the absolute instantaneous velocity of *b* is represented by the vector *bb'*. To find the velocity of *K*, determine the absolute velocity of *c* and its components along and perpendicular to *bc*, as *cg* and *ch*. In § 39, it is shown that the velocity of each particle is the resultant of a rotation about some point in the body and the absolute velocity of this center of rotation. In the body being considered, that is, the link *B*, the velocities of *b* and *c* perpendicular to *bc* may be considered as being due to rotation about some point on *bc* extended. Since the velocities due to rotation are proportional to the distances of the particles from the center of rotation, this point can be located by drawing a line through *e* and *h* intersecting *bc* extended at *p*. The point *p* has a velocity along *cb* equal to *bf* equal to *cg*, but has no velocity due to rotation.

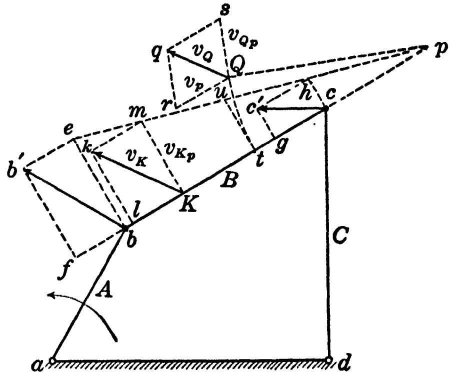


FIG. 39

Now the velocity of K is the resultant of the velocity of p and the velocity relative to p, or

$$v_K = \text{vector sum } (v_p + v_{Kp}) = Kk = \text{vector sum } (Kl + Km),$$

where Kl is equal to bf. Similarly, the velocity of Q, any point not on the line bc, is the resultant of the velocity of p and the velocity of Q relative to p. The magnitude of the velocity relative to p is found by drawing the arc Qt with p as a center and drawing tu perpendicular to pcb intersecting phe at u. The direction of this velocity is perpendicular to pQ. Then Qs, equal to tu and perpendicular to pQ, represents the velocity of Q relative to p. The absolute velocity of Q is

$$v_Q = \text{vector sum } (v_p + v_{Qp}), \qquad v_Q = Qq = \text{vector sum } (Qr + Qs).$$

The angular velocity of the link B can now be found from the relation

$$v = r\omega, \quad \text{or} \quad \omega = \frac{v}{r},$$

since the velocities of b, K, c, and Q due to rotation about p are known.

The angular velocity of B is

$$\omega = \frac{v_{bp}}{bp} = \frac{be}{bp} = \frac{v_{Kp}}{Kp} = \frac{Km}{Kp} = \frac{v_{Qp}}{Qp} = \frac{tu}{tp} \text{ etc. radians per second,}$$

where the velocity vectors, be, Km, etc., are measured in feet per second, and the distances, bp, Kp, etc., are measured in feet.

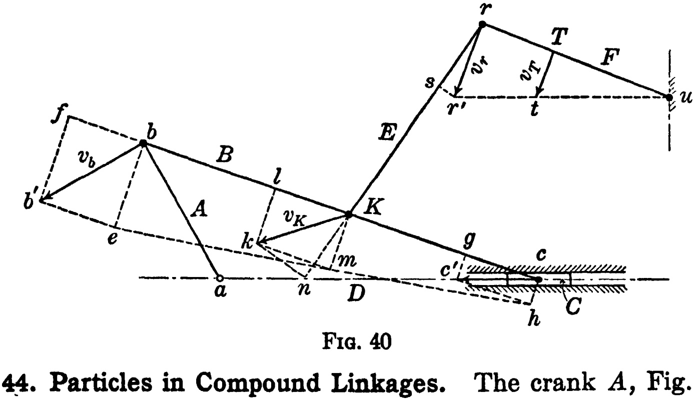


FIG. 40

44. Particles in Compound Linkages. The crank A, Fig. 40, drives the slide C as well as the crank F, which is connected to

B by the link E. The absolute velocity of b is represented by the vector bb'.

The construction for the absolute velocity of C was discussed in § 41, and that for the absolute velocity of K was discussed in the preceding article.

To determine the velocity of r, resolve Kk into two components along and perpendicular to Kr. Draw rs equal to Kn. The absolute velocity of r will be along rr', which is perpendicular to ur. Draw sr' perpendicular to Kr. Sr' intersects rr' at r'; therefore rr' represents the absolute velocity of r to the same scale that bb' represents the absolute velocity of b.

The construction for the absolute velocity of T was discussed in § 40.

45. Particles in Sliding Members. In Fig. 41, the crank A is the driver, rotating about a and causing C to oscillate about c. Let the vector bb' represent the velocity of b on the crank A.

Since the point b may represent particles on links A and C or a particle on the slide B, and since the velocities of these three particles must be considered, we will indicate the particle on link A by the symbol b_A, the particle on link C by b_C and on the slide B by b_B.

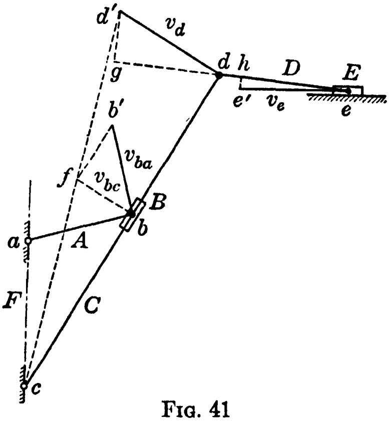


FIG. 41

The link C is constrained to oscillate about c. The velocity of b on C, that is, the particle b_C, must be perpendicular to cd, and it must also be equal to the component velocity of b_A in the same direction. Resolve bb' into components bf and fb' perpendicular and parallel to cd. The vector component fb' represents the velocity at which the slide is moving along C, and bf represents the velocity at which b_A is moving perpendicular to cd. The vector bf also represents the velocity at which b_C is moving perpendicular to *cd*. Draw *cfd'*. Draw *dd'* perpendicular to *cd*, intersecting *cfd'* at *d'*. The vector *dd'* represents the velocity of *d*. Resolve the velocity of *d* into components *dg* and *gd'* along and perpendicular to the link *ed*. Draw *eh* equal to *dg*. The velocity of E is parallel to the guide in the direction *ee'*. Draw *he'* perpendicular to *ed*, intersecting *ee'* at *e'*. The vector *ee'* represents v_E, the velocity of the slide E.

46. Velocity-Space Diagrams. In each of the preceding discussions, the instantaneous velocity of each of the particles was found for some particular position of the linkage. In many cases, it is desirable to study the velocities at various positions, together with the changes in velocity and the rapidity of these changes as the linkage passes through a complete cycle. For this purpose a velocity diagram is plotted.

In Fig. 42, a block A slides along a straight guide D. When the block is at A, let the vector Aa represent its velocity, to scale. When the block is at A_1, let the vector A_1a_1 represent its velocity. As A moves along a straight path, velocity vectors drawn for various positions of A will all fall on the same line. To separate these vectors, rotate them into positions perpendicular to their actual directions. Thus Aa rotated into the position Ab represents the magnitude of the velocity of A, but not its direction. Similarly A_1b_1, A_2b_2, etc. represent the magnitudes of the velocity of A when A is at A_1, A_2, etc. A line drawn through b, b_1, b_2, etc. indicates in the form of a graph the velocity of A as a function of its position as A moves along the guide. This diagram is called a ***velocity-space diagram.***

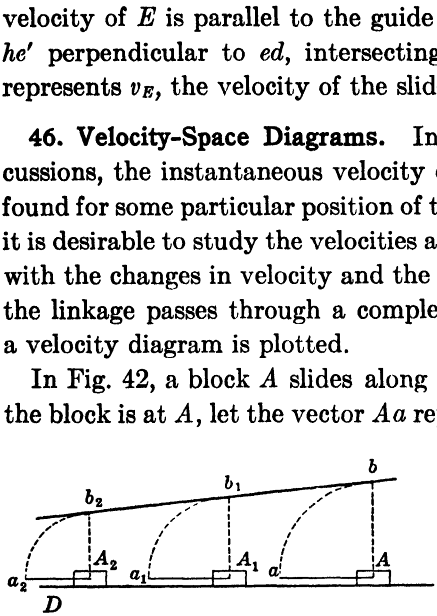


FIG. 42

47. Polar Diagrams. When the particle A is moving on a curved path such as *cad* in Fig. 43, the velocities may be plotted on radial lines using as a base line the path of the particle; or, if the path is circular, the center of rotation may be used as the center of the velocity curve. It is best to use the path of the particle as the ***base line.*** This diagram is called a ***polar velocity-diagram.*** The magnitude of the velocity of A at any instant is measured by the radial distance between the base line and the velocity curve. Its direction of course is tangent to the path, or base line.

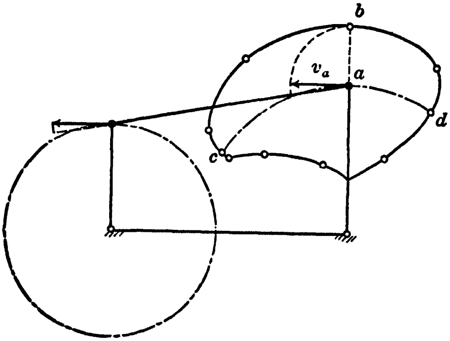


FIG. 43

Sometimes the base line of the polar diagram is straightened, and the velocities are plotted perpendicular to the straight line. Figure 44 shows the polar velocity diagram of Fig. 43 re-plotted with the base line straight. This is called a ***rectilinear velocity-diagram.***

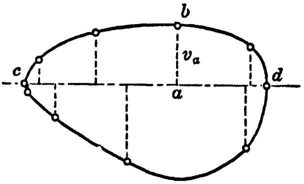


FIG. 44

48. Negative Velocity. If the particle is reciprocating or oscillating, it may have different velocities when passing the same point on its path in opposite directions. It is customary to call the velocities during the forward stroke positive and those during the return stroke negative. Usually, positive velocities are plotted above the base line in rectilinear diagrams, and negative velocities below the base line. In polar diagrams, positive velocities are plotted radially out from the base line, and negative velocities radially in.

Figure 45 shows the velocity diagrams for particles b, d, and e, if we assume that the crank A rotates with uniform velocity.

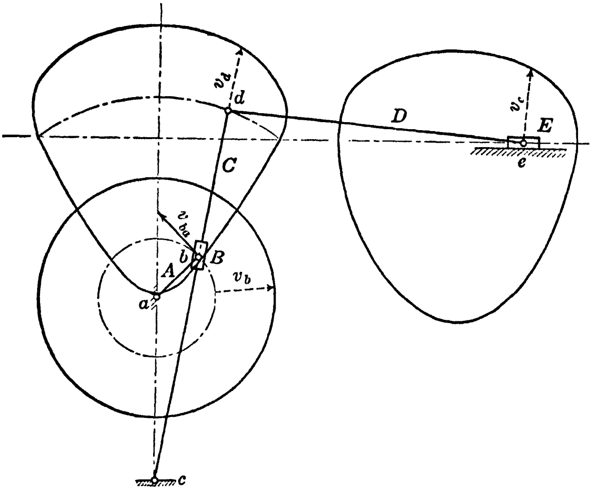


FIG. 45

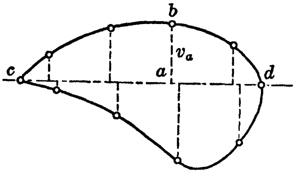


FIG. 46

49. Velocity-Time Diagrams. If the base line is laid off in equal time units instead of equal space intervals, the diagram is called a *velocity-time* diagram. Figure 46 is the same as Fig. 44, except that in Fig. 46 the abscissas are time intervals, while in Fig. 44 the abscissas are space intervals, the velocity-time diagram, Fig. 46, being replotted from the velocity-space diagram Fig. 44.

PROBLEMS

1. A particle A is moving horizontally to the right with an absolute velocity of 20 ft. per min. A particle B, rigidly attached to A, moves up and to the right at an angle of 60° with the horizontal with a velocity relative to A of 30 ft. per min. What is the absolute velocity of B?

2. An automobile is traveling horizontally at 25 miles per hr. The piston in one of its cylinders is moving downward over the cylinder walls at the rate of 1200 ft. per min. What is the absolute velocity of the piston?

3. The boom of a locomotive crane is being raised at the rate of 2 radians per min. The whole crane is moving forward with a velocity

of 30 ft. per min. When the boom is at an angle of 30° with the horizontal in the direction of the crane movement, determine the absolute velocity of the end of the boom, if the boom is 12 ft. long.

4. One end, *A*, of the connecting-rod of an engine is moving toward the right on a horizontal line. The other end, *B*, is moving up and to the right at an angle of 45°. The connecting-rod is 3 ft. long and makes an angle of 15° with the horizontal.

(a) What is the angular velocity of the rod about the end *A*, in radians per sec.?

(b) What is the absolute angular velocity of the rod?

5. A ladder 15 ft. long is resting against a vertical wall with its lower end 5 ft. from the wall and on a horizontal floor. The bottom end is pulled away from the wall at the rate of 5 ft. per sec.

(a) Determine the velocity of the top of the ladder.

(b) Determine the velocity of the middle point of the ladder.

(c) Determine the velocity of the point on the ladder which is moving the slowest.

6. An automobile having tires 32 in. in diameter is traveling 40 mi. per hr. to the right.

(a) Determine the absolute velocity of the center of the wheel in ft. per sec.

(b) Determine the velocity relative to the center, of a point on the periphery of the tire.

(c) Determine graphically the absolute velocity of a point on the periphery of the tire when it is on a radial line through the center making an angle of 60° with the horizontal.

7. The center of a wheel is moving to the right with a velocity of 5 ft. per sec. A particle *B* 1½ in. from the center is revolving clockwise about the center 800 times per min.

(a) When *B* is horizontally to the right of the center, determine its absolute velocity.

(b) When *B* is directly below the center, determine the absolute velocity.

8. A wheel rotates clockwise at 50 r.p.m. about its center, and the center moves to the right 4 ft. per sec. Determine the velocity of a point 12 in. from the center when

(a) directly to the right of the center.

(b) directly below the center.

(c) on a radial line making 210° with the horizontal.

9. A horizontal plank is supported at one end on a plane making an angle of 60° with the horizontal, and rests on a smooth peg 6 in. in diameter whose center is 5 ft. from the smooth plane. The plank is 8 ft. long.

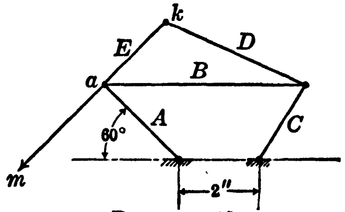
(a) Determine the velocity of the free end of the plank when one end slides down the plane with a velocity of 10 ft. per min.

(b) Which point on the plank has the least velocity? Determine the velocity of this point.

Scale of velocity 1 in. = 5 ft. per min.

PROBLEMS 10 TO 15.

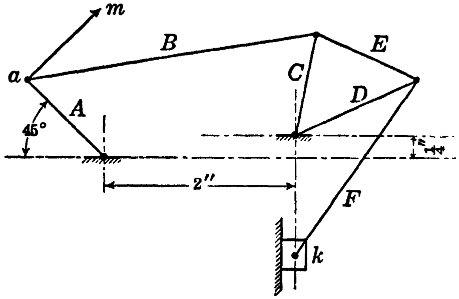
In each of the following figures, the velocity of the point *a* is given as shown. Determine the velocity of the point *k*, stating the magnitude and direction in each case.



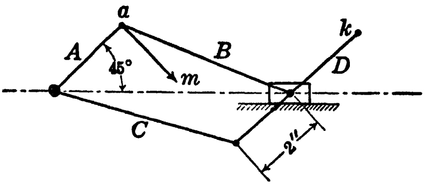
PROBLEM 10

10. am = 1 in. = 5 ft. per sec.
A = ¾ in.,
B = 2 in.,
C = 1¼ in.,
E = 1 in.,
D = 1¼ in.

11. am = 1.25 ft. per sec.
A = 1 in.,
B = 2½ in.,
C = ¾ in.,
D = 1 in.,
E = ¾ in.,
F = 2 in.



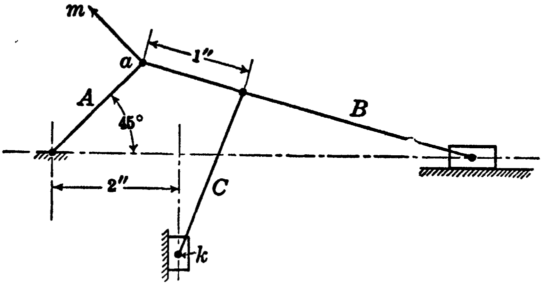
PROBLEM 11



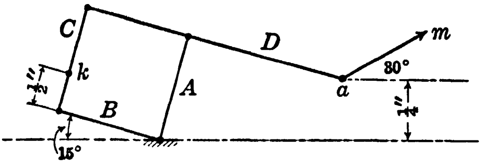
PROBLEM 12

12. am = .75 ft. per sec.
A = ⅞ in.,
B = 3 in.,
C = 2¾ in.,
D = 4 in.

13. am = 750 ft. per min.
A = 2 in.,
B = 5 in.,
C = 5 in.



PROBLEM 13

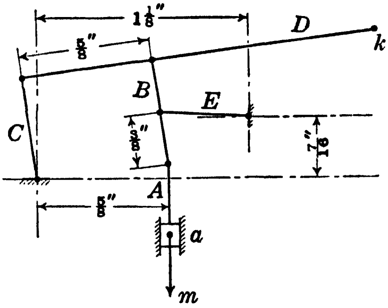


PROBLEM 14

14. am = 80 ft. per min.
A = 1¼ in. = C,
B = 2 in.,
D = 5 in.
A parallel to C.

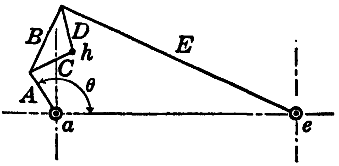
15. am = ½″ per sec.
A = 2½ in.,
B = ¾ in.,
C = ⅞ in.,
D = 3¾ in.,
E = ⅝ in.

E makes an angle of 15° with the horizontal.



PROBLEM 15

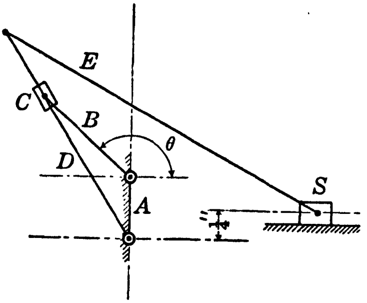
16. In the figure, the crank is rotating at 10 r.p.m. Determine the velocity of h and of the mid-point of E.



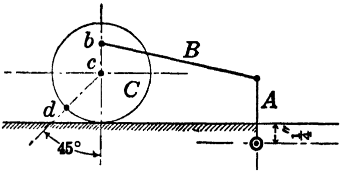
PROBLEM 16

A = 1 in.,
B = ¾ in.,
C = D = ½ in.,
E = 2¾ in.,
ae = 2½ in.,
$\angle\theta$ = 135°

17. In the figure, the crank B is rotating at 6 r.p.m. Determine the velocity of the slide S when the angle θ is 120°.
$A = \frac{3}{4}$ in., $B = 1\frac{1}{2}$ in., $D = 3$ in., $E = 4\frac{1}{2}$ in.



PROBLEM 17



PROBLEM 18

18. In the figure, the crank A is rotating clockwise at 5 r.p.m.
(a) Determine the velocity of d when A is vertical.
(b) Determine the velocity of d when A has rotated 45° to the right.
$A = 1\frac{1}{2}$ in., $B = 3$ in., $C = 2$ in. diam., $bc = \frac{1}{2}$ in.

19. A body sliding along an oiled surface is brought to rest by the combined action of friction and an external force, so that the equation of motion is $s = 225t - 3t^3$, where s is measured in feet and t in seconds.
(a) Determine the time required for the body to come to rest.
(b) Plot the space-time diagram and the velocity-time diagram.
Scale of time: 1 in. = $\frac{1}{2}$ sec.
Scale of velocity: 1 in. = 100 ft. per sec.
Scale of space: 1 in. = 200 ft.

20. A hoist in a mine starts from rest at the bottom of the shaft and accelerates upward for 5 sec. with uniform acceleration of 8 ft. per sec. per sec. It next travels with uniform velocity for 6 sec., and then is stopped with uniform retardation in 10 sec.
(a) Determine the total depth of the mine.
(b) Plot the velocity-time diagram for this hoist.
Scale of velocity: 1 in. = 25 ft. per sec.
Scale of time: 1 in. = 20 sec.
(c) Plot the velocity-space diagram for this hoist.
Scale of velocity: 1 in. = 25 ft. per sec.
Scale of space: 1 in. = 500 ft.

21. In the figure of Problem 13, plot the velocity-space diagram for the slide k when the crank A makes a complete revolution. Velocity of the point a to be represented by a vector 1 in. long.

CHAPTER VI

INSTANTANEOUS CENTERS

50. Definition of Instantaneous Centers. In § 20, it is shown that the plane motion of any body at a particular instant can be duplicated by a simple rotation about a point called the instantaneous center of rotation.

The instantaneous centers of a mechanism are the points in the fixed link (extended if necessary) about which each of the other links rotates or tends to rotate at the instant under consideration. Therefore, the *fixed link will contain an instantaneous center of rotation for each link in a given mechanism.* The point in a moving link about which any other moving link rotates or tends to rotate is sometimes called a ***common velocity point.*** *Each moving link of a given mechanism will therefore have a common velocity point with each of the remaining moving links.*

Both instantaneous centers and common velocity points are included under the general term ***centros*** and will be discussed later under that heading.*

This chapter will deal only with the instantaneous centers of a mechanism and the method used to determine the velocities of various points in a mechanism by their use.

51. Location of the Instantaneous Center. The velocity of any particle of a rotating body is at any instant tangent to its path, or perpendicular to the radius joining the particle and the center of rotation. If the directions of the velocities of any two particles of the body not moving parallel to each other are known, the instantaneous center can be located. In Fig. 47, the velocities of the particles A and B have directions as indicated by the lines Aa and Bb. The

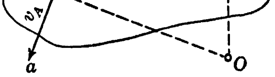


FIG. 47

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* See Chapter 7 for a discussion of centros.

instantaneous velocity of A can be produced by rotation about any point on AO drawn perpendicular to Aa. Similarly, the instantaneous velocity of B can be produced by a rotation about any point on BO drawn perpendicular to Bb. Since O is a point on both lines, a rotation of the body about O will duplicate the instantaneous velocity of both particles and of all other particles on the body.

52. Instantaneous Centers of a Four-Bar Linkage. The two cranks A and C of Fig. 48 rotate or oscillate about the fixed centers a and d. Then a is always the instantaneous center for the crank A, and d is always the instantaneous center of the crank C. The directions of the velocities of two points on the link B are known, since b and c are moving perpendicular to ab and dc respectively. Extend ab and dc to intersect at O. These lines are perpendicular respectively to the velocities of b and c, and therefore their intersection O is the instantaneous center of the link B, as demonstrated in the preceding paragraph.

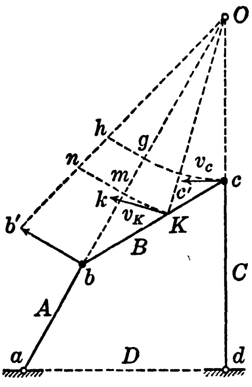


FIG. 48

In § 17, it is shown that the linear speeds of two particles on a rotating body vary directly as their distances from the center of rotation. It follows in Fig. 48 that

$$\frac{\text{linear speed of } b}{\text{linear speed of } c} = \frac{Ob}{Oc}.$$

If bb' represents the velocity of b to scale, the velocity of any particle on the link, as c or K, can be found as follows. With a radius equal to OK, draw the arc Km, intersecting Ob at m. Draw mn perpendicular to Ob, intersecting Ob' at n. The line mn represents the magnitude of the velocity of K, since

$$\frac{mn}{bb'} = \frac{Om}{Ob} = \frac{OK}{Ob}.$$

The direction of the velocity of K is perpendicular to OK. Draw Kk perpendicular to OK and equal to mn. The vector Kk represents the velocity of K.

Similarly cc', equal to gh, represents the velocity of c.

53. Crank and Crosshead. It is shown in § 24 that a sliding member is equivalent to a crank of radius equal to the radius of curvature of the path of the sliding member. The equivalent crank is always perpendicular to the path of the sliding member. In Fig. 49, the equivalent crank of the slide C is perpendicular to ac and is of infinite length. Draw this equivalent crank as cO extended. Extend the crank center line ba to intersect cO at O. The point O is the instantaneous center of the link or connecting rod B.

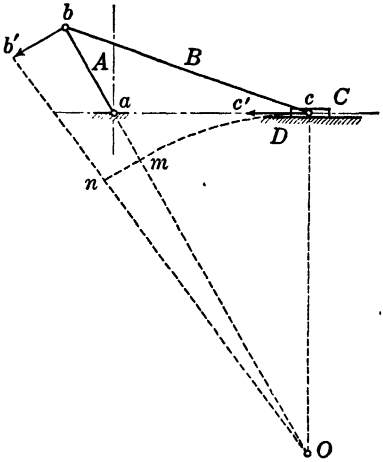


FIG. 49

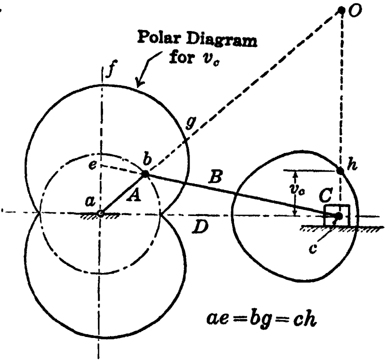


FIG. 50

The velocity of b being known, the velocity of c can be found by the method of § 52. A special method adapted to this linkage is illustrated in Fig. 50. Locate the instantaneous center of B at O. Draw af perpendicular to ca. Extend bc to intersect af at e.

$$\frac{\text{linear speed of } b}{\text{linear speed of } c} = \frac{Ob}{Oc} = \frac{ab}{ae}.$$

Therefore, if *ab* represents to scale the linear speed of *b*, then *ae* represents the linear speed of *c* to the same scale. The distance *ae* found for different positions of the crank can be plotted radially from the crank-pin circle to form the polar velocity-diagram of the crosshead. The distance *ch*, equal to *ae*, for different positions of the crosshead, may also be plotted at *c* perpendicular to the guide to form a rectangular velocity-diagram for the crosshead *C*.

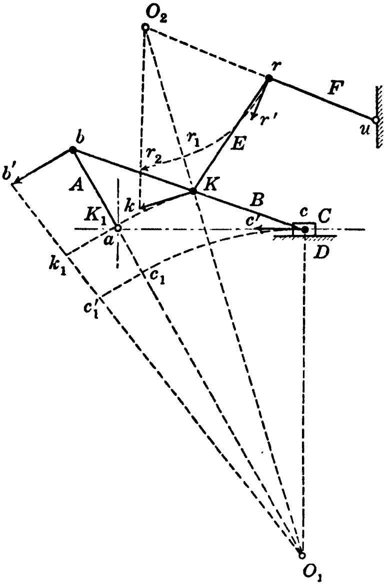


FIG. 51

54. Compound Linkages. In Fig. 51, let the vector *bb'* represent the velocity of *b* on the driving crank *A*. In § 45, the velocity of *r* on the oscillating crank *F* was found by vector components. This velocity can be found by using the instantaneous centers of the links *B* and *E*.

Extend *ba* and draw cO_1 perpendicular to the path of *c*, and intersecting *ba*, extended, at O_1, the instantaneous center of *B*. Draw O_1b'. With a radius O_1K, draw the arc KK_1. Draw K_1k_1 perpendicular to O_1b, intersecting O_1b' at k_1. Draw Kk perpendicular to O_1K and equal to K_1k_1. The vector Kk represents the velocity of the particle, *K*, since

$$\frac{v_K}{v_b} = \frac{OK}{Ob} = \frac{OK_1}{Ob} = \frac{K_1k_1}{bb'} = \frac{Kk}{bb'},$$

or

$$v_K = \frac{v_b}{bb'} \times Kk = Kk.$$

Extend *ur* to intersect O_1K, extended, at O_2. The point O_2 is the instantaneous center of the link *E*. Draw O_2k. With O_2r as a radius draw the arc rr_1. Draw r_1r_2 perpendicular to O_2K

and intersecting O_2k at r_2. Draw rr' perpendicular to ur and equal to r_1r_2. The vector rr' represents the velocity of the particle r.

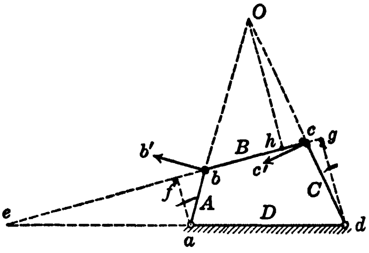


FIG. 52

55. Angular Velocity Ratio of Two Cranks of a Four-Bar Linkage. In Fig. 52, let bb' represent the velocity of the particle b in the crank A. Locate the point O, the instantaneous center of the link B. Extend ad and bc to intersect at e. Draw Oh, af, and dg perpendicular to bc. Then

$$\frac{\text{linear speed of } b}{\text{linear speed of } c} = \frac{Ob}{Oc}.$$

But the linear speed of b is $ab \times \omega_A$, and the linear speed of c is $dc \times \omega_C$; therefore

$$\frac{Ob}{Oc} = \frac{ab \times \omega_A}{dc \times \omega_C}, \qquad \text{and} \qquad \frac{\omega_A}{\omega_C} = \frac{Ob \times dc}{ab \times Oc}.$$

Since the pairs of triangles Ohb and afb, Ohc and dgc, and efa and egd are similar, we have

$$\frac{Ob}{ab} = \frac{Oh}{af}, \quad \frac{dc}{Oc} = \frac{dg}{Oh}, \quad \frac{dg}{af} = \frac{ed}{ae}.$$

Hence, substituting, we find

$$\frac{\omega_A}{\omega_C} = \frac{Oh}{af} \times \frac{dg}{Oh} = \frac{dg}{af} = \frac{ed}{ae}. \tag{25}$$

Therefore *the angular velocities of two cranks connected by a rigid link are inversely as the lengths of the perpendiculars from their fixed centers to the center line of the connecting link or its extension,* and also *inversely as the distance from the fixed centers to the intersection of the center line of the connecting link and the line passing through the fixed centers.*

56. Determination of Angular Velocity Ratio. For example, consider the drag-link mechanism of Fig. 53. In the position

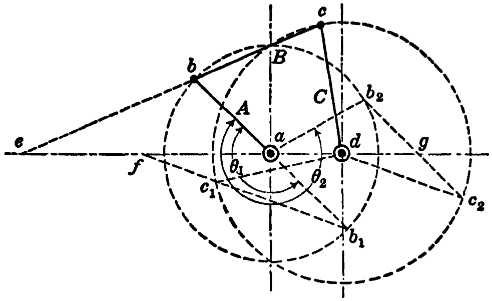


FIG. 53

shown, the center line bc of the connecting link B, extended, intersects da, extended, at e. The distances ae and de can be measured and their ratio found. If the angular velocity of the driving crank A is uniform and considered to be unity, then the ratio ae/de will be the angular velocity of the follower crank C, or

$$\omega_C = \frac{ae}{de} \times \omega_A \text{ radians per second.}$$

Similarly when the crank A has moved through angles θ_1 and θ_2, respectively, we have

$$\omega_C = \frac{af}{df} \times \omega_A \text{ radians per second,}$$

and

$$\omega_C = \frac{ag}{dg} \times \omega_A \text{ radians per second.}$$

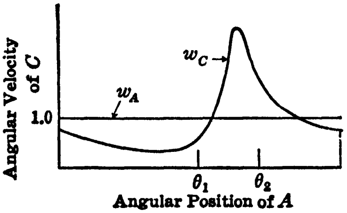


FIG. 54

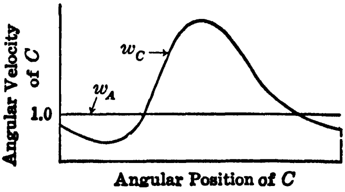


FIG. 55

The angular velocities thus obtained may be plotted to scale against the angular position of A as shown in Fig. 54. For some purposes, it may be desirable to plot the angular velocities of C against the angular position of C as shown in Fig. 55.

Since the linear speed v_c of c is equal to ω_C times the crank radius dc, the curves also represent the linear speeds of the particle c, to the proper scale. Thus

$$\text{scale of } v_c = \text{scale of } \omega_C \times \text{length } dc.$$

57. Angular Velocity Ratio with Sliding Members. In Fig. 56, assume that the crank A is the driver, rotating uniformly in a counter-clockwise direction. The mechanism shown is a four-bar linkage and must have two cranks and two connecting links, real or imaginary. Of these the fixed connecting link D and the crank A are readily seen. The sliding member B and its guide (the lever C) are equivalent to an imaginary link perpendicular to the guide, passing through b, and of infinite length. This imaginary link is the second connecting-rod of the four-bar linkage and must be considered as being pinned to the crank A at b. This rod is shown as E in the figure. The other end of this imaginary connecting-rod must be pinned to the second crank. The fixed point of the second crank is d, and since the imaginary connecting-rod is infinite in length, this crank must be parallel to E. This crank is shown as F in the figure. The four-bar linkage then consists of the cranks A and F, the fixed link D, and the connecting rod E. The imaginary rod F is in fact part of the lever C since it is always perpendicular to C and oscillates on the same fixed center d.

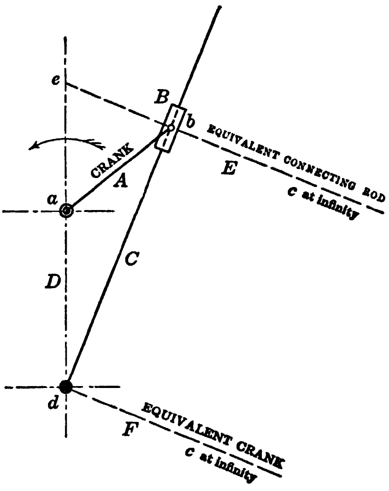


FIG. 56

To find the angular velocity-ratio extend ad and bc to intersect at e. We then have for the angular velocity-ratio of the two cranks

$$\frac{\omega_F}{\omega_A} = \frac{ae}{de} = \frac{\omega_C}{\omega_A},$$

whence

$$\omega_C = \frac{ae}{de} \times \omega_A,$$

from which the angular velocity of C can be found when the angular velocity of A is known.

58. Instantaneous Center of Rolling Bodies. If a cylinder is rolled along a guiding surface without slipping, as in Fig. 57, the point O, where contact occurs with the guiding surface, is at this instant stationary, and the whole body may be considered as turning about O as an instantaneous center.

The velocity of any particle on the rolling body, as B, is perpendicular to the line OB, joining the particle with the instantaneous center. In rolling bodies, such as automobile or locomotive wheels, the velocity of the center or axis of the wheel is usually known, and from this the velocity of any other particle can be found. The magnitude of the velocity of B can be found from the relation

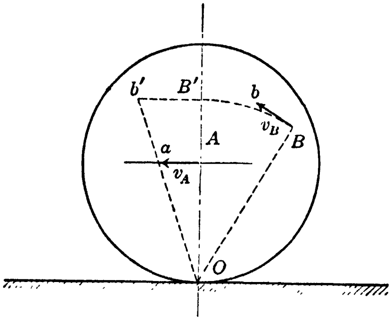


FIG. 57

$$\text{linear speed of } B = \text{linear speed of } A \times \frac{OB}{OA} \cdot$$

Extend the line OA, and draw Oa extended. With OB as a radius, draw the arc BB'. Draw $B'b'$ perpendicular to OA, and intersecting Oa extended at b'. Draw Bb perpendicular to OB and equal to $B'b'$. The vector Bb represents the velocity of the particle B.

59. Application to Trailers. The principle of instantaneous centers is utilized in making truck trailers so that they will follow the path traversed by the truck or other hauling medium. If the two pairs of wheels in Fig. 58 are connected as in an ordinary wagon or trailer, the front pair pivots on p, the center of the front axle, when turning a corner. The instantaneous center of the truck body is located at *O*, and since the front wheels *A* and *B* are farther from *O* than the respective rear wheels, the front end moves on an arc of larger radius than the rear end. A string of trailers of this type could not be made to follow the leading truck in a sinuous path.

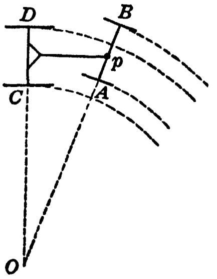


FIG. 58

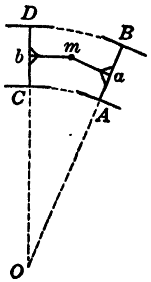


FIG. 59

By moving the pivot point to *m*, as in Fig. 59, equidistant from the two axles, making *am* equal *mb*, the wheels *A* and *C* will move on the same circular arc, and *B* and *D* will move on the same arc. A trailer made so as to use the pair of wheels *CD* as front wheels and to use the pivot point *m* as the coupling pin will then follow the leading truck or trailer wheels *AB* in any path.

Figure 60 shows a string of such trailers. The rear end of each trailer body can be supported on wheels free to turn on a vertical axis such as is used on three-wheel hand trucks.

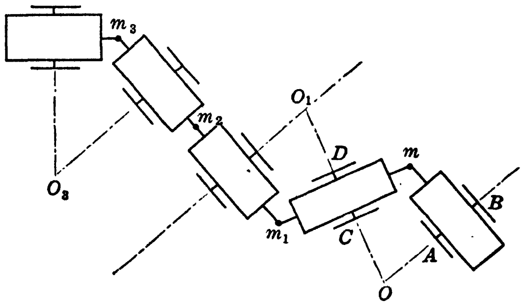


FIG. 60

The trailers or trucks will swing more easily if the rear supporting wheels, or castors, are made so that the vertical axis of rotation passes slightly to one side of the horizontal axis of rotation of the wheel. With this arrangement, the wheel rolls behind the vertical supporting spindle or shaft; and when this shaft moves sidewise, the wheel turns about a vertical axis so that it can roll instead of slide in the new direction. This principle is used in mounting the front wheels of bicycles, tricycles, and automobiles, as well as in mounting the trailing wheel of trailers, invalid chairs, and similar wheeled devices.

60. Velocities by Transference. In many complex linkages, considerable time may be saved if the velocities are transferred by a method different from those previously described in this chapter. Since this method is a modification of the method of instantaneous centers, much of the detail of solution in the following examples is omitted.

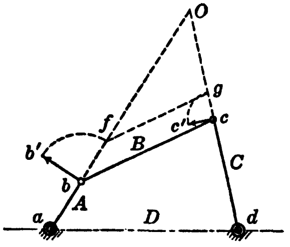


FIG. 61

61. Four-Bar Linkage. In Fig. 61, let the vector bb' represent the velocity of b. The velocity of c can be found as follows: Extend the line ab. With a radius equal to bb', draw the arc $b'f$ intersecting ab, extended, at f. Extend the line dc. Draw fg parallel to bc, intersecting dc, extended, at g. Draw cc' perpendicular to dc and equal to cg. The vector cc' represents the velocity of c.

PROOF. To prove this construction, locate the instantaneous center of B. This instantaneous center is at O. Then

$$\frac{\text{velocity of } b}{\text{velocity of } c} = \frac{Ob}{Oc} = \frac{bf}{cg} = \frac{bb'}{cc'},$$

since the intercepts between the base of a triangle and a line parallel to it are proportional to the sides of the triangle.

62. Crank and Crosshead. In Fig. 62, let the vector bb' represent the velocity of b. The velocity of c is to be found as follows: Extend the line ab. Draw cO perpendicular to the path of C, intersecting ab, extended, at O. With bb' as a radius, draw the arc $b'f$ intersecting ab, extended, at f. Draw fg parallel to bc intersecting cO at g. Draw cc' tangent to the path of c and equal to cg. The vector cc' represents the velocity of C.

To find the velocity of any particle, as K, in the link B, draw OK intersecting fg at m. With Km as a radius, draw the arc *mk*. Draw *Kk* perpendicular to *OK* and intersecting the arc *mk* at *k*. The vector *Kk* represents the velocity of *K*.

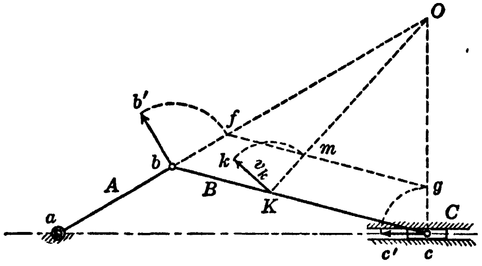


FIG. 62

63. Velocities in Compound Linkages. In Fig. 63, let the vector *bb'* represent the velocity of *b*. It is desired to determine

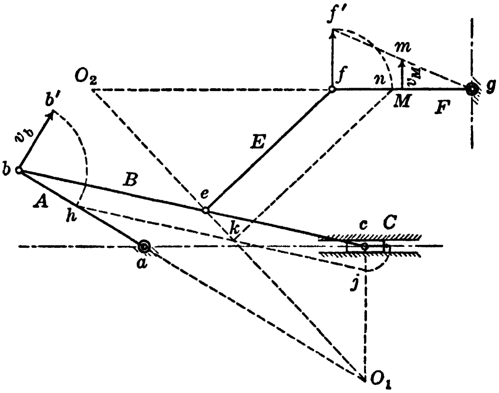


FIG. 63

the velocity of *M* in the crank *F*. Extend *ab* and draw cO_1 perpendicular to the path of *C*, intersecting *ab*, extended, at O_1. With *bb'* as a radius, draw the arc *b'h*. Draw *hj* parallel to *bc*. Draw O_1e intersecting *hj* at *k*. The line *ek* represents the magnitude of the velocity of *e*. Draw *kn* parallel to *ef* intersecting *gf* at *n*. Draw *ff'* perpendicular to *gf* and equal to *fn*. The vector *ff'* represents the velocity of *f*. Draw *f'g*. Draw *Mm* perpendicular to *fg* and intersecting *f'g* at *m*. The vector *Mm* represents the velocity of the particle *M*.

64. Second Compound Linkage. As a second example of compound linkages, consider Fig. 64, in which the crank A is the driver, and b_A has a known velocity represented by the vector bb'. The velocity of E is to be found as follows: Since B is a sliding member, it is equivalent to an imaginary link perpendicular to cd and of infinite length. The common-velocity point for this link and the link C will be at an infinite distance from b. Using the method of transference developed in the preceding articles, extend ab, and with bb' as a radius, draw the arc $b'g$. Draw gh parallel to the imaginary link, that is, perpendicular to cd, intersecting cd at h. Draw bk perpendicular to cd and equal to bh. The vector bk represents the velocity of the particle b_C. Draw ck extended. Draw dd' perpendicular to cd and intersecting ck extended at d'. Extend cd and draw en perpendicular to the path of E. With dd' as a radius, draw the arc $d'm$. Draw mn parallel to de and intersecting en at n. Draw ee' tangent to the path of E and equal to en. The vector ee' represents the velocity of E.

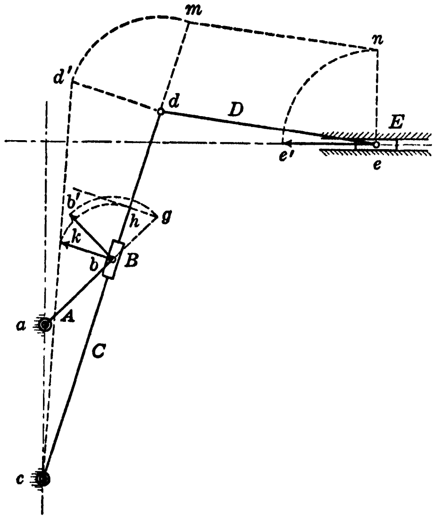


FIG. 64

Note that the velocity of b_C was obtained more easily by other methods.

PROBLEMS

1. A simple four-bar linkage consists of two cranks 1¼ in. and 2 in. long. The connecting rod is 3 in. long, and the distance between fixed centers is 2½ in. Locate the instantaneous center of the connecting rod, and trace the locus of the instantaneous center for a complete rotation of the 1¼ in. crank.

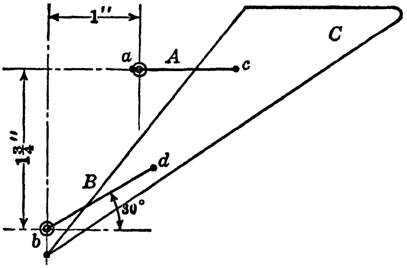
2. The crank-and-pitman motion of a horizontal bull-dozer for heading bolts consists of a crank 2 in. long, a pitman-rod (or con-

necting-rod) 8 in. long, and a slide moving on a horizontal center line which passes through the center of rotation of the crank. Locate the instantaneous center of the pitman-rod and trace the locus of this point for a complete revolution of the crank *A*.

3. A drag-link mechanism consists of a crank *A*, 1 in. long, a crank *B*, 1½ in. long, and a connecting-rod *C*, 1¾ in. long. The distance between fixed centers is ½ in. Trace the locus of the instantaneous center of *C*.

4. In the figure, which point of the triangular plate *C* has zero velocity? What is the point called?

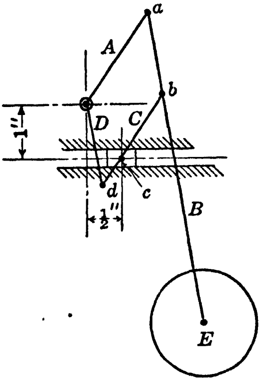
$cd = 1\frac{1}{4}$ in.,
$ac = 1$ in.,
$bd = 1\frac{1}{4}$ in.



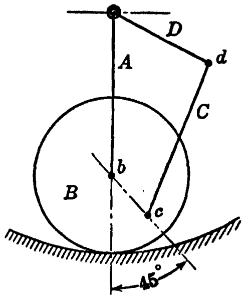
PROBLEM 4

5. In the figure, locate the instantaneous center of *B* and *C*. What is the direction of motion of the center of the wheel *E*?

$A = 2$ in. $= C$, $B = 6$ in., $D = 1\frac{1}{2}$ in. $= ab$, $dc = \frac{1}{2}$ in.



PROBLEM 5

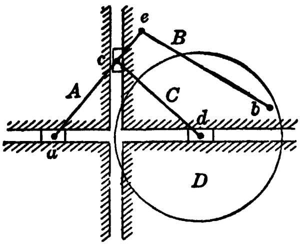


PROBLEM 6

6. In the figure, locate the instantaneous centers of *A*, *B*, and *C*, and indicate the direction of the motion of point *c*.

$A = 1\frac{1}{2}$ in.,
$D = 1$ in.,
$C = 1\frac{1}{2}$ in.,
$B = 1\frac{1}{2}$ in. diam.,
$bc = \frac{1}{2}$ in.

7. In the figure, locate the instantaneous centers of A and B when the angle cad is 45°. The disk D rotates on the center d.



PROBLEM 7

$ae = 1\frac{7}{8}$ in.,
$be = 2$ in.,
$dc = 1\frac{1}{2}$ in.,
$ac = 1\frac{3}{8}$ in.,
$db = 1$ in.

8. A simple four-bar linkage consists of a crank A, $1\frac{1}{4}$ in. long, a crank C, $1\frac{3}{4}$ in. long, and a connecting-rod, 3 in. long. The distance between the fixed centers is $2\frac{1}{2}$ in. Crank A makes an angle of 45° with the line joining the fixed centers.

(a) Locate the instantaneous centers of A, B, and C.

(b) If the velocity of a point on A located $\frac{3}{4}$ in. from the fixed center is represented by a vector 1 in. long, find the velocity of the pin joining A and B.

(c) Determine the velocity of a point on B, 1 in. from the pin joining A and B, and 3 in. from the pin joining B and C.

9. A four-bar linkage has cranks A and C, each $2\frac{1}{2}$ in. long, with their fixed centers $4\frac{1}{2}$ in. apart. The connecting-rod B is $4\frac{1}{2}$ in. long. A rotates uniformly at 100 r.p.m. Determine the angular velocity of C in r.p.m. when A makes an angle of 60° with the line of centers, and with the connecting-rod crossing the line of centers.

10. The two cranks, A and C, of a four-bar linkage have their fixed centers 4 in. apart. $A = 2$ in., $C = 3$ in. Connecting-rod $B = 5$ in. A is rotating uniformly at 90 r.p.m. Determine the angular velocity of C in *radians per second* when A is perpendicular to the line of centers.

11. A wheel 4 ft. in diameter is rolling along a horizontal plane. If the absolute velocity of the center of the wheel is represented by a vector $1\frac{1}{4}$ in. long, find graphically the velocities of the top of the wheel, the point on the rim directly in front of the center of the wheel, and the point mid-way between the center and rim on a radial line of the wheel which makes an angle of 45° back of the vertical center line.

12. Two cranks have a distance between fixed centers of 2 in. Crank A is $1\frac{1}{2}$ in. long, crank C is $1\frac{3}{4}$ in. long, and the connecting link is 2 in. long. Crank A is the driver.

(a) Determine the position of A when the linkage is on dead points.

(b) Determine the position of A for maximum angular velocity of C, and state the ratio of the angular velocity of C to that of A.

13. In a horizontal steam engine, the crosshead travels on a center line passing 3 in. above the center of the crank shaft. The crank is 1 ft. long, and the connecting-rod is 4 ft. long.

(a) Determine the crank angle at the dead center positions.

(b) Determine the ratio of times required for the forward and return strokes.

(c) Determine the position of the crank for maximum velocity of the crosshead.

14. In a gas-engine, the crank is 1½ in. long. The connecting-rod is 5 in. long, and the crank makes an angle of 120° with the path of the piston-pin. Determine the velocity of the middle point of the connecting-rod when the crank rotates at a speed of 1000 revolutions per min.

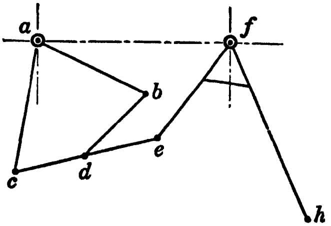
15. A ladder 10 ft. long rests against a vertical wall with the bottom 3 ft. from the wall. If the bottom of the ladder is moved along the floor toward the wall at the rate of 1 ft. per sec., what will be the velocity of the top of the ladder, the middle point of the ladder, and the point having the least velocity?

16. A horizontal plank 6 ft. long rests on a smooth plane which makes an angle of 45° with the horizontal, and on a smooth peg 6 in. in diameter. The shortest distance from the center of the peg to the smooth plane is 3½ ft. The end of the plank slides down the plane with a velocity of 1 ft. per second. Determine completely the velocity of the opposite end of the plank, the middle point, and the point having the least velocity.

17. In the figure, $ab = 2$ in., $bd = 1\frac{1}{2}$ in., $ac = 2\frac{1}{4}$ in., $cd = 1\frac{1}{8}$ in., $ce = 2\frac{1}{4}$ in., $ef = 2\frac{3}{8}$ in., $fh = 3\frac{3}{4}$ in., and $af = 3\frac{1}{4}$ in. The levers ab and ac turn freely on the fixed center a. The angle $efh = 60°$.

(a) When the angle afh is 120°, determine the velocity of the point b, assuming that the velocity of the point h is represented by a vector 1 in. long.

(b) Same as (a) when the angle $afh = 60°$.



PROBLEM 17

18. In Problem 8, determine the angular velocity ratio between the cranks A and C.

19. In Fig. 51 make $A = 1\frac{1}{2}$ in., $B = 5$ in., $bK = 2$ in., $F = 1\frac{3}{4}$ in. The point u is located 4 in. to the right and $1\frac{3}{4}$ in. above the point a. Determine the angular velocity ratio between B and F when the angle bac is 120°.

20. An oscillating steam engine has a crank 2 in. long and a piston rod (or connecting-rod) 6 in. long. The cylinder is pivoted so as to oscillate about the center point of the piston stroke. If the crank makes 200 r.p.m., what is the angular velocity of the cylinder when the crank makes an angle of 60° with the line joining the crank shaft and center of oscillation of the cylinder?

21. In the figure of Problem 5, the slide moves with a velocity represented by a vector 1 in. long. Determine the velocity of points *a* and *d*, and the center of the wheel *E*.

22. In the figure of Problem 6, the angular velocity of *A* is 2 radians per sec. What is the angular velocity of the wheel *B*? Using the angular velocity just determined, find the angular velocity of *D*.

23. In the figure of Problem 7, the center of the wheel moves to the right with a velocity of 2 ft. per sec. Determine the velocity of the points *a* and *b*.

24. An automobile has a wheel base of 110 in., and the wheel treads are 4 ft. 8½ in. apart. When the front wheels are turned at an angle of 15°, locate the instantaneous center. If a two-wheeled trailer is used, where should the trailer be pinned to the automobile in order to have perfect trailing, with the trailer wheels 6 ft. behind the rear wheels?

25. A horizontal steam engine running at 200 r.p.m. has a stroke of 3 in. and a connecting-rod 6 in. long. The center-line of the piston path passes ¼ in. below the fixed center of the crank. Plot the velocity-space curve for the piston to a scale of 1 in. = 100 ft. per min.

CHAPTER VII

CENTROS AND COMBINED VELOCITY METHODS

65. Centers of Relative Motion. With reference to the simple four-bar linkage as shown in Fig. 65, if the link D is fixed, then the points a and d are fixed centers of rotation, and also the instantaneous centers for the cranks A and C. The point b is the common point of A and B, that is, it is a point which may be considered as being a part of either A or B. If B is the fixed link, then b is the center of rotation of A. When D is the fixed link, b is still the point in B about which A rotates relative to B, although B is no longer stationary. The point b is also the point in A about which B rotates relative to A. Similarly the point c is the center of rotation of B and C relative to each other.

If ab and dc are extended to intersect at O_1, the instantaneous center of B is found. This point, O_1, is also the center of rotation of B relative to D, and of D relative to B. Similarly bc and ad, extended, intersect at O_2, which is the center of rotation of A relative to C, and of C relative to A.

66. Centros. The point in one body about which another body is rotating, or tends to rotate, is called a ***centro***. It is also *the point, common to two bodies, which has the same absolute velocity in both bodies*. This point is the *center of relative motion between the two bodies.*

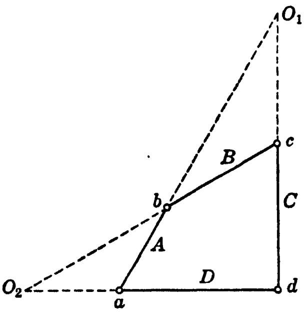


FIG. 65

If the absolute velocity of the common-velocity point of two bodies is equal to zero, the common-velocity point becomes an instantaneous center. It is evident from the construction of a mechanism that all connecting points between the fixed link and the other links have zero velocity and are therefore known as instantaneous centers.

Thus the point O_2, in Fig. 65, is the centro of A and C, since it is the point in A (enlarged or widened to include O_2) about which C is at this instant rotating relative to A. If both A and C are enlarged and connected by a pin at O_2, their instantaneous motions relative to each other are duplicated by a rotation of either one about O_2.

Point a is the centro of A and D. Point c is the centro of B and C. Point O_1 is the centro of B and D, etc.

In § 52, the point O_1 is called the instantaneous center of B. An instantaneous center is simply a special case of centros in which one of the links is fixed. Thus if the link D is fixed, a is the instantaneous center of A, d is the instantaneous center of C, and O_1 is the instantaneous center of B. However, if A is the fixed link, O_2, a and b are the instantaneous centers. In either case the motions of the links relative to each other are the same. Points such as a, b, c, and d where the links are pinned together, are called ***permanent centros.***

In dealing with centros, it is customary to designate each body, or link, by one capital letter and the centros by the two letters representing the links to which the point is common. The centros are designated by lower case letters as ab, bc, etc. It is immaterial whether the centro is called ab or ba, since the motion of A relative to B is the same as the motion of B relative to A, except that the sense is reversed.

67. Number of Centros in Any Linkage. Since every pair of links has a centro, the number of centros for any linkage is

$$\text{No. of centros} = \frac{N(N-1)}{2}, \tag{26}$$

where N is the total number of links.

68. Kennedy's Theorem. Kennedy's Theorem states that the *centros of any three bodies having plane-motion relative to each other must lie in the same straight line.*

Figure 66 shows three bodies, A, B, and C, with A and B joined by a pin joint at ab, and with A and C joined by a pin joint at ac. The centros of A and B and of A and C must be at the pin joints ab and ac. To locate the centro of B and C, it is necessary to locate a point common to both links and with the same absolute velocity in each.

Select any point, *p*, common to both *B* and *C*. Since *B* is rotating about *ab*, the point *p* in *B* is moving perpendicular to *ab–p* or in the direction *pd*. Since *C* is rotating about *ac*, the point *p* in *C* is moving perpendicular to *ac–p* or in the direction *pe*. The lines *pd* and *pe* are not coincident, and the point *p* does not have the same velocity in the two bodies *B* and *C*. The centro of *B* and *C* must be such a point that *pd* and *pe* coincide. Since *pd* and *pe* are perpendicular to *ab–p* and *ac–p* respectively, it is evident that *ab–p* and *ac–p* must coincide. This condition is possible only when *p* lies on the straight line joining *ab* and *ac*. Therefore, the three centros of *A*, *B*, and *C* lie in the same straight line, as stated in Kennedy's Theorem.

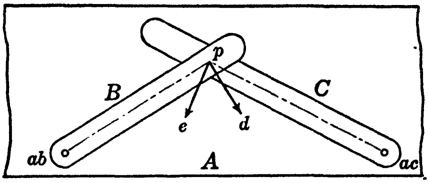


FIG. 66

The centro *bc* is now known to be on the line *ab–ac*, but its exact location cannot be determined since the links, as shown, are not constrained to any definite relative motion. If the connecting link between *B* and *C* is known, the exact location of *bc* can be determined.

69. Location of All the Centros of a Constrained Linkage. First locate all of the permanent centros, and then by the use of Kennedy's Theorem locate the remaining centros.

In Fig. 67, the permanent centros are *ad*, *ab*, *bc*, *cd*, *be*, *ef*, and *df*.

The centro *cd* is at infinity on a line through *bc* and perpendicular to the path of the slide *C*. This must be true, since the equivalent crank which would cause *C* to move in a straight line is of infinite radius.

To locate the centro *ac*, two combinations of three links each, both containing the links *A* and *C*, are necessary. The combination *A*, *B*, and *C*, and the combination *A*, *D*, and *C*, are suitable, since the centros *ab* and *bc* are known, and the centros *ad* and *dc* are known. The centro *ac* must lie on the straight line drawn through *ab* and *bc*, and on the straight line drawn through *ad* and *dc*. These lines intersect at *ac*, the centro of *A* and *C*.

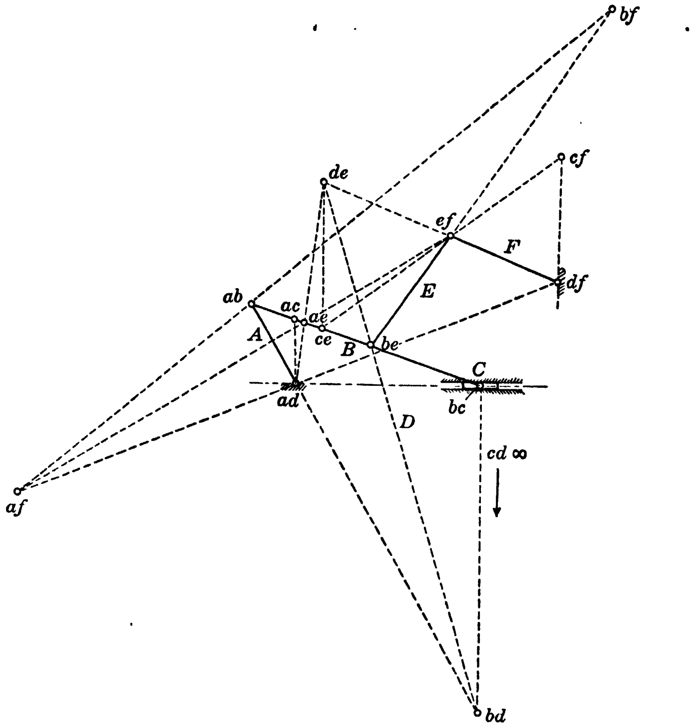


Fig. 67

The other centros are located in the same manner. Note that in the selection of two pairs of centros, each pair must have a common letter, and the odd letters of each pair must be the same as the centro to be located. For example, *ab* and *bc*, used with *ad* and *dc*, locate *ac*, and *bd* and *be*, used with *df* and *ef*, locate *de*.

70. Polygon Check Diagram. Any attempt to locate all the centros of a compound linkage without some form of guide may be extremely tedious, unless extreme luck is experienced in the selection of the sequence in which the centros are located. The simplest guide is a polygon having as many sides as there are

links, including sliding members and fixed links. Each corner of the polygon is lettered with the symbol letter of one of the links.

To use this diagram, join with a heavy line the corners which represent pairs of links whose centros are known. The beginning of the check polygon for Fig. 67 is Fig. 68. To locate any new centro, draw a line joining two corners of the polygon in such a way that this new line is the common side of two triangles, the other sides of which are made up of lines representing centros that are already known. If *A* and *C*, Fig. 68, are connected, the line *AC* is the common side of the triangles *ABC* and *ADC*. The centros *ab*, *bc*, *ad*, and *dc* represented by the sides of these triangles are known, and therefore can be used to locate the centro *ac* represented by the common side *AC*. As the centros are located, draw in lines on the check polygon to represent them.

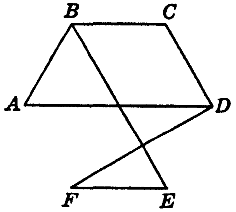


FIG. 68

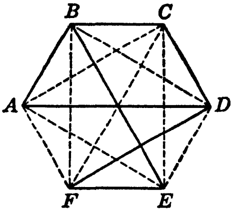


FIG. 69

The remaining centros are located in the same manner. When all of the centros have been located, every corner of the check polygon will be connected to every other corner as in Fig. 69.

71. Velocities by the Use of Centros. FOUR-BAR LINKAGE. In Fig. 70, the link *D* is fixed, and the crank *A* rotates. Let the vector *k–l* represent to scale the velocity of the particle *k* on the crank *A*.

TO DETERMINE VELOCITIES OF PARTICLES IN CONNECTING LINKS. In order to determine the velocity of the particle *m* in the link *B*, it is necessary to locate the centros of the two links *A* and *B* and the stationary link *D*. These centros are *ad*, *ab*, and *bd*. Centros *ad* and *bd* are points on the fixed link and are, therefore, stationary. Centro *ab* is the common velocity point of links *A* and *B*, and is moving. The velocity of the moving centro must be determined first.

The velocities of two particles on the same rotating body are to each other as their distances from the center of rotation. Therefore to determine the velocity of *ab*, draw *ad–l* extended.

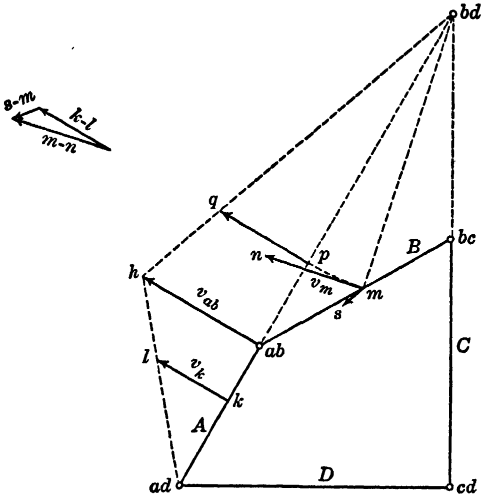


Fig. 70

Draw *ab–h* perpendicular to *ad–ab,* intersecting *ad–l* at *h.* The vector *ab–h* represents the velocity of *ab.*

To determine the velocity of any particle on *B,* as *m,* draw *bd–h,* and, with *bd–m* as a radius, draw the arc *m–p.* Draw *p–q* perpendicular to *bd–ab,* intersecting *bd–h* at *q.* Draw *m–n* perpendicular to *bd–m* and equal to *p–q.* The vector *m–n* represents the velocity of *m.*

Note that this construction is the same as that used in § 52 with instantaneous centers. This is to be expected since *bd* is a stationary centro and therefore the instantaneous center of the link *B.*

The link *B* is rotating about *ab,* relative to the crank *A.* The velocity of *m* relative to *k,* on *A,* is found if the velocity vector of *k* is subtracted from the velocity vector of *m.* The vector *m–s* represents this vector difference and, therefore, represents the velocity of *m* relative to *k.* The velocity of *k* relative to *m* is represented by the reverse of this or the vector *s–m.*

72. Velocities of Particles in Opposite Links. In Fig. 71, the velocity of particle *k* on *A* is known, and the velocity of particle *m* on *C* is desired. It is necessary to locate the centros of *A* and *C* and the fixed link *D*. These are *ad*, *cd*, and *ac*; *ad* and *cd* being stationary. The centro *ac* is a point on *A*, and the velocity of *k* on this link is known. With *ad–k* as a radius, draw the arc *k–f*. Draw *f–g* perpendicular to *ad–ac* and equal to *k–l*. Draw *ad–g* extended. Draw *ac–h* perpendicular to *ad–ac*, intersecting *ad–g*, extended, at *h*. The vector *ac–h* represents the velocity of the centro *ac*. The point *ac* is also on the link *C* and has the same velocity as *ac* on *A*. With a radius *cd–m*, draw the arc *m–s*. Draw *cd–h*. Draw *s–t* perpendicular to *cd–ac*, intersecting *cd–h* at *t*. Draw *m–n* perpendicular to *cd–bc* and equal to *s–t*. The vector *m–n* represents the velocity of the particle *m*.

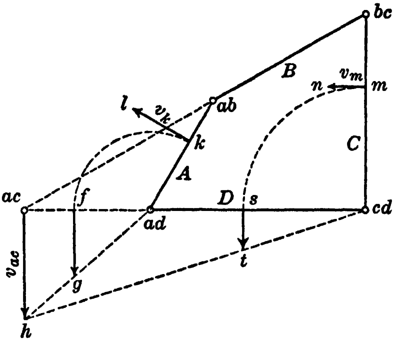


FIG. 71

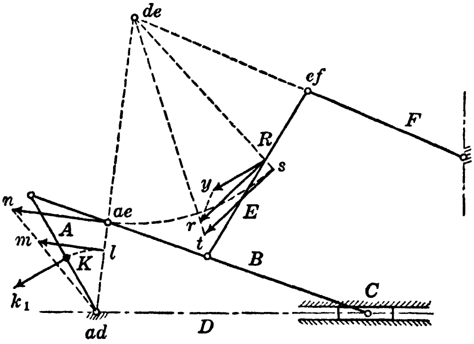


FIG. 72

73. Velocities in Compound Linkages. In Fig. 72, the velocity of the particle *K* on the crank *A* is known, and the velocity of the particle *R* on the link *E* is desired. It is necessary to locate the centros of the links *A*, *E*, and *D*. These are *ad*, *de*, and *ae*. Centros *ad* and *de* are stationary. The centro *ae* is on the link *A*, and the velocity of the particle *K* on this link is known. By construction similar to that used in the preceding cases, the vector *ae–n* is found to represent the velocity of *ae*. The centro *ae* is also a point on the link *E*. Draw a line through *de* and *R*. With a radius *de–ae* draw the arc *ae–s*. Draw *s–t* perpendicular to *de–s* and equal to *ae–n*. Draw *de–t*. Draw *Rr* perpendicular to *de–R*, intersecting *de–t*, extended, at *r*. The vector *R–r* represents the absolute velocity of the particle *R*.

The velocity of *R* relative to *K* is found if the velocity vector of *K* is subtracted from the velocity vector of *R*. The vector *y–r* represents the velocity of *R* relative to *K*.

74. Velocities in a Second Compound Linkage. As an example of a different type of compound linkage, consider the linkage of

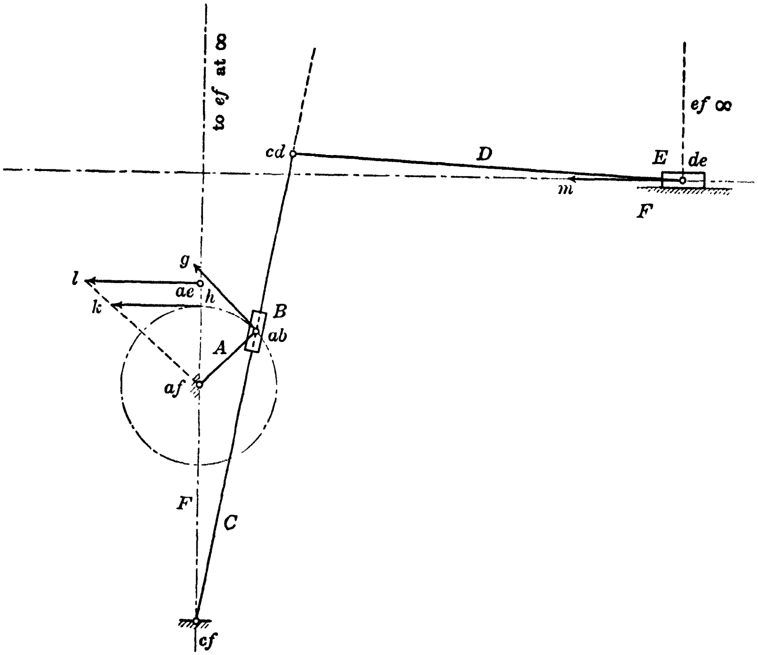


Fig. 73

Fig. 73. Let the crank *A* be the driver rotating at a known velocity. Let the vector *ab–g* represent the velocity of the crank pin joining the crank *A* to the slide *B*. To determine the velocity of the slide *E*, it is necessary to locate the centros of the links *A*, *E*, and *F*. These are *af*, *ef*, and *ae*. The centros *af* and *ef*

are stationary. The centro *ae* is on the crank *A*, and the velocity of the point *ab* on this crank is known. With a radius *af–ab* draw the arc *ab–h* intersecting *af–ae* at *h*. Draw *h–k* perpendicular to *af–ae* and equal to *ab–g*. Draw *af–k*, extended. Draw *ae–l* perpendicular to *af–ae* intersecting *af–k* extended at *l*. The vector *ae–l* represents the velocity of *ae*. The centro *ae* is also on the link *E*, which is revolving about the centro *ef* at infinity. Since *ef* is at infinity, the velocity of all particles on *E* must be equal and parallel. Therefore draw *de–m* equal and parallel to *ae–l*. The vector *de–m* represents the velocity of the slide *E*.

75. Combination of Methods. From the discussions of the various methods of transferring velocities, it is evident that for some linkages one method is considerably easier than others. For other linkages, two or more methods may be used with equal ease, rapidity, and accuracy.

It is, therefore, a matter of individual preference as to which method is used. Combinations of two or more methods may be desirable. In the remainder of this work, the authors will use whichever method seems to be easiest, clearest, and most accurate. In cases where a particular type of linkage has to be analyzed frequently, considerable ingenuity can be applied in developing special constructions applicable to that linkage only. An example of this is the method developed in § 53 for the crank and crosshead.

76. Solutions for Miscellaneous Velocity Problems. The preceding articles on velocities discuss the various ways of finding the velocity of one part of a linkage when the velocity of another part is known. Different methods are taken up in the different chapters with examples to illustrate their applications. With each method, the same figures were used for the examples in order to allow the student to compare the different methods.

The following examples are worked out in order to give some idea of the method of procedure in tracing the velocities through various types of linkages. Some of the linkages are abstract, while others are taken from mechanisms in use. The actual mechanisms used, however, are not always to scale, for it is necessary to exaggerate in order to bring out the method of tracing velocities in small scale drawings.

As a rule, the student finds it easier to combine several of the methods in working out a complicated linkage, and this practice has been followed in several of the examples. The method of tracing velocities by the use of centros is not used in the solution of any of the following examples, for centros were sufficiently discussed in the first part of this chapter. The method of procedure in the use of centros is the same, regardless of the linkage, and the following examples could readily be solved by that method.

77. Example 1.

GIVEN. The four-bar linkage *ABCD* in Fig. 74, where *A* is a crank rotating about the fixed point *a*. The block *C* is pivoted to the frame on a horizontal center line through *c*. The connecting link *B* is free to slide in the block *C*. The point *e* is fastened to the link *B*. The absolute velocity of *b* is represented by the vector bb_1.

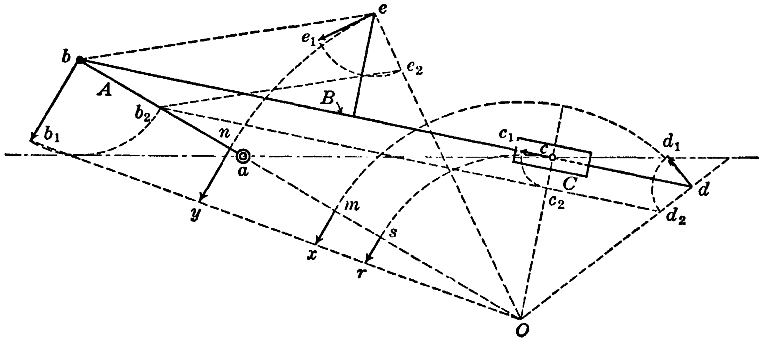


FIG. 74

REQUIRED. The absolute velocities of *d*, *e*, and the point on *B* that is moving the slowest.

CONSTRUCTION. The point *b* is on the link *A* as well as on the link *B*, since it is the connecting point between the two links, and its absolute velocity is the same, regardless of the link considered. Extend *ab* until it intersects at the point *O* a line through *c* perpendicular to *bc*. The point *O* is the instantaneous center of *B*. The velocity of *b* on *B* is bb_1. Revolve the point *d* about *O* to *m* in the line *bO*, and draw a perpendicular to *bO* at *m*, intersecting the line from b_1 to *O* at *x*. Then *mx* is the magnitude of the velocity of *d*. Draw dd_1 perpendicular to *Od* and equal to *mx*. The vector dd_1 represents the absolute velocity of *d* in magnitude and direction. In like manner, ee_1 represents the absolute velocity of *e*, for *e* is a point on the link *B* and is, therefore, revolving about *O* at this instant.

The point on *B*, which is moving with the slowest velocity, may be found if a perpendicular is dropped to *bd* from *O*, and the point is thus located at *c*. In other words, the point on a link which has the smallest velocity is the point nearest the center about which the link is rotating. The velocity of *c* is cc_1 and was found in the same manner as the velocity of *d*.

78. Alternate Construction for Example 1. Locate the instantaneous center *O* as before, and draw the lines *dO* and *eO*. Rotate bb_1 to bb_2. The point b_2 is on the line from *b* to the instantaneous center *O*. Draw a line from b_2 parallel to *bd* until it intersects *dO* at d_2. Then dd_2 represents the magnitude of the velocity of *d*. Revolve dd_2 about *d* to dd_1, perpendicular to *dO*. The vector dd_1 represents the absolute velocity of *d* in magnitude and direction, to the same scale that bb_1 represents the velocity of *b*. In like manner, cc_1 equal to cc_2 represents the velocity of *c*, the point of least velocity.

To find the velocity of *e*, draw a line from b_2 parallel to a line joining *b* to *e* until it intersects *eO* at e_2. Then ee_2 represents the magnitude of the velocity of *e*. Revolve ee_2 about *e* into a line perpendicular to *eO*, finding the vector ee_1 which represents the absolute velocity of *e* in magnitude and direction.

The velocity of any other point in *B* or *B* expanded could be found in a manner similar to the above construction.

79. Example 2.

GIVEN. The linkage *ABCDE* shown in Fig. 75, where *a*, *c*, and *e* are fixed points, and *B* and *D* are blocks sliding in grooves in the curved link C. The velocity of b is represented by the vector bb_1.

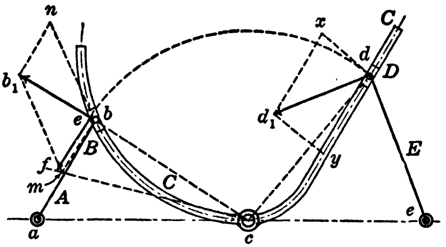


FIG. 75

REQUIRED. The absolute velocity of D and the velocity of D relative to its guides.

CONSTRUCTION. The absolute velocity of the pin point b in the slide B is perpendicular to the crank A, and the absolute velocity of the pin point d in the slide D is perpendicular to the crank E. Both b and d will have component velocities due to the rotation of the link C about c. These component velocities will be the same as the absolute velocities of the points b and d when considered as points in the link C. Resolve the vector bb_1 into components, one to determine the velocity of b about c, and the other to determine the velocity of b relative to its guides. The vectors bm and bn represent these component velocities. The vector bm is perpendicular to a straight line joining b and c, and the curvature of the guides of B has no effect on this component. Since the guides of B are curved, bn is drawn tangent to the path at the point b. The vector bm represents the velocity of b about c, and the velocity of any other point in the link, as d, about c can be found by the proportion: velocity of d about c is to bm (the velocity of b about c) as dc is to bc. The vector dx represents the velocity of d about c, and the construction for finding dx, equal to ef, is indicated in the figure. The absolute velocity of the slide D will be along a line perpendicular to de at the point d. The absolute velocity of the point d in the slide D will have a component velocity along the guides (representing the velocity of d relative to the link C) in addition to the component dx. From x draw xd_1 parallel to the guides, cutting dd_1 at d_1. The vector dd_1 represents the absolute velocity of the point d in the slide D. When the parallelogram is completed, dy_1, which represents the velocity of d relative to the guides, is found.

80. Example 3.

GIVEN. The velocipede shown in Fig. 76, rolling on the track *T*, consists of two wheels with a frame *F* to hold the wheel axes, *a* and *b*, fixed relative to each other and to the propelling linkage *A–B–C*. The driving wheel is keyed to a crank *C* turning about *b*, and *B* is a link between the link *A*, which is pivoted at *a*, and the crank *C*. The vector aa_1 represents the velocity of the point *a* relative to the track.

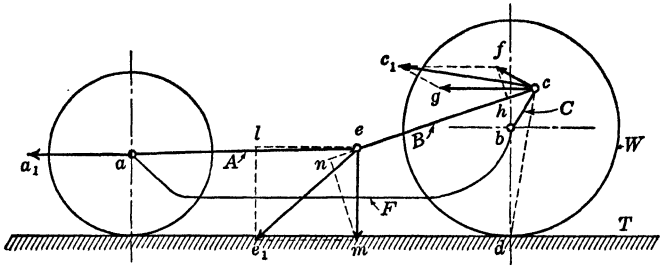


FIG. 76

REQUIRED. The absolute velocity of the point *e*.

CONSTRUCTION. The vector aa_1 represents the velocity of the point *a* to the left; therefore every point on the velocipede will be moving to the left with the same velocity. The absolute velocity of *e* is composed of the velocity to the left and the velocity due to rotation of *e* about *a*. The oscillation of *e* about *a* is caused by the crank *C*. The point *c* has a velocity to the left represented by *cg* equal to aa_1. The absolute velocity of *c* is along the line cc_1, perpendicular to *cd*, since *d* is the instantaneous center of the wheel *W*. In addition to the component to the left, the absolute velocity of *c* will have a component perpendicular to *bc* because of the rotation of *C* about *b*. Draw a line through *g* perpendicular to *bc*, cutting the line cc_1 at c_1, and complete the parallelogram. Then cc_1 will represent the absolute velocity of *c*, and *cf* will represent the velocity with which *c* is turning relative to *b*. Resolve the velocity *cf* into two components, *ch* and *hf*, one along *ce* and one perpendicular to *ce*, and make *en* equal to *ch*. Erect a perpendicular to *ec* at *n*, and erect a perpendicular to *ea* at *e*. These perpendiculars will intersect at *m*, and the vector *em* will represent the velocity of *e* relative to *a*. The point *e* has a velocity to the left, represented by *el* equal to aa_1, and a velocity relative to *a*, represented by *em*. Complete the parallelogram, and ee_1 will represent the absolute velocity of *e* in magnitude and direction.

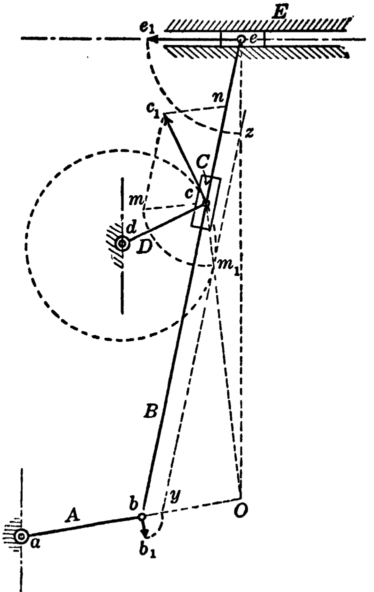


Fig. 77

81. Example 4.

Given. The Averbeck shaper linkage *ABCDE* in the position shown in Fig. 77, where *a* and *d* are fixed points, and the block *C*, pivoted to the link *D*, slides on the link *B*. The crank *D* is the driver, and the absolute velocity of *c* is represented by the vector cc_1.

Required. The absolute velocity of *E*.

Construction. The links *A*, *B*, and *E* form a crank and slide-block linkage. The instantaneous center, *O*, of *B* can be found if the arm *ab* is produced until it intersects, at *O*, a line drawn through *e* perpendicular to the path of the slide-block *E*. Every point in *B* is rotating about *O* at this instant. The velocity of c_B about *O* will be along a line perpendicular to *cO* through *c*. Resolve cc_1 into two components, *cn* and *cm*, one along the path of the slide, *be*, and the other perpendicular to *cO*. The vector *cm* represents the absolute velocity of the point c_B about *O*. The vector *cn* represents the velocity of the slide-block relative to *B*. Rotate *cm* to cm_1, and draw a line through m_1 parallel to *be*, cutting *bO* at *y* and *eO* at *z*. Then *ez* is the magnitude of the absolute velocity of *e*. Rotate *ez* to ee_1. The vector ee_1 represents the absolute velocity of *E* in magnitude and direction. In like manner, the vector bb_1 represents the absolute velocity of *b* in magnitude and direction.

82. Alternate Construction for Example 4. The absolute velocity of *E* may be found by composition and resolution of velocities as follows: The velocity of *E* may be considered as being due to a rotation of *be* relative to *b*, combined with a translation of *b* due to *b* rotating about *a*. In Fig. 78, the absolute velocity of *c* due to rotation about *d* is known and is represented by the vector cc_1. If the point *b* is considered as fixed, the vector cc_1 can be resolved into two components, *cn* and *cm*, one along and one perpendicular to the link center line *be*. The component *cn* is the velocity of the slide relative to *B*, and *cm* is the velocity of the point c_B in *be* relative to *b*. Draw a line from *b* through *m* until it intersects a perpendicular to *be* through *e* at e_2. Then the vector ee_2 will represent the velocity of *e* due to rotation about *b*. The point *e* also has motion due to the crank *A;* therefore assume *c* to be a fixed point. Since the velocity of *b* due to rotation about *a* is not known, assume any velocity for *b* as *bh*, and draw *bh* perpendicular to *ab* at *b*. Break up the velocity *bh* into two components, *bg* and *bf*, along and perpendicular to *be*. The component *bf* is the velocity of *b* due to rotation about the point *c*. Draw a line from *f* through *c* cutting ee_2 at *x*. Then *ex* represents the velocity of *e* about *c*, and *ey* equal to *bg* represents the velocity of *e* along the link *B* if *bh* represents the velocity of *b*. Complete the parallelogram at *e*, and *ez* will represent the direction of the velocity of *e* due to the motion of *b*. The absolute velocity of *e* is along the center line of the slide-block. Therefore draw a line through e_2 parallel to *ez*, cutting the center line at e_1. Then the vector ee_1 represents the absolute velocity of *E*.

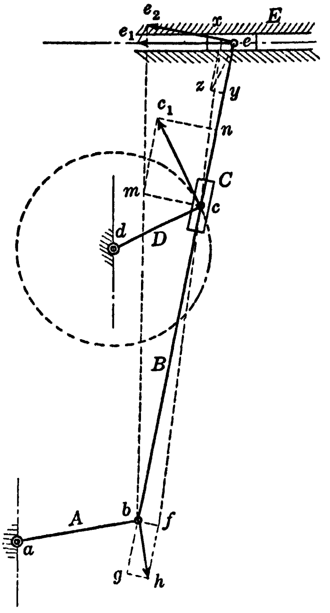


FIG. 78

83. Example 5.

GIVEN. Watt's sun and planet motion is shown in Fig. 79. This mechanism was once used as a substitute for the crank on steam-engines. The circles *W* and W_1 represent pitch circles of gears. The wheel *W* rotates about *a*, and the wheel W_1, fixed to the link *B*, does not rotate on the center *b* relative to *B*. *B* is a connecting link between the slide *C* and the wheels. The vector *kl* represents the velocity of the point *k* about *a*.

REQUIRED. The absolute velocity of the slide *C*.

CONSTRUCTION. Rotate *kl* to k_1l_1, and draw a line from *a* through l_1 cutting a perpendicular to *ab* through *m* at *n*. Then *mn* represents the absolute velocity of *m*. Since the wheel W_1 does not rotate with respect to the link *B*, the point *m* is a point on the link *B* expanded. *O* is the instantaneous center of the link *B*, and the velocity of the point *m* on the link *B* is represented by *mn*. Draw a line from *O* through *n*, cutting a perpendicular to *Oab* through *b* at b_1. Then bb_1 represents the absolute velocity of *b*. By the construction shown using composition and resolution of velocities, cc_1 is found to represent the absolute velocity of *C*.

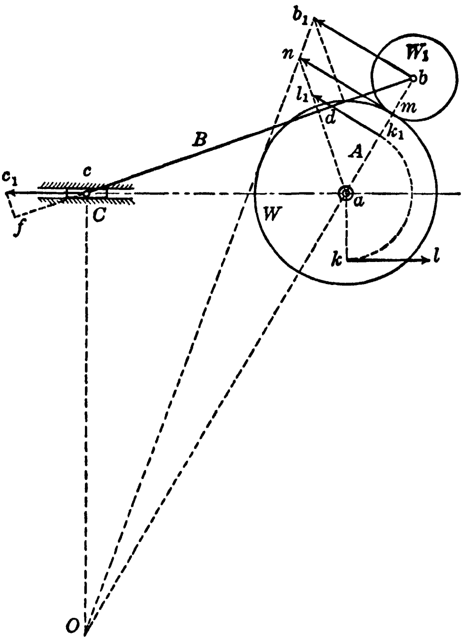


FIG. 79

84. Example 6.

GIVEN. The wheels *X*, *Y*, and *Z*, in Fig. 80, rolling on each other without slipping. The wheel *X* is fixed to the shaft *A*, and the wheel Z is free to rotate about the shaft A. The wheel Y is in contact with both wheels X and Z. The vector Bb represents the absolute velocity of a point on the rim of the wheel Z.

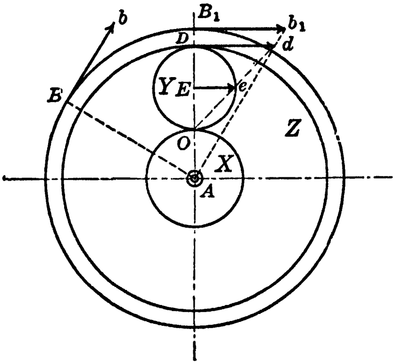


FIG. 80

REQUIRED. The absolute velocity of the center E of the wheel Y.

CONSTRUCTION. Rotate Bb to B_1b_1, and draw a line from b_1 to A. A perpendicular to AB_1 at D will intersect Ab_1 at d. Then the vector Dd represents the absolute velocity of D on the wheel Z. The wheel Y is in contact with the wheel X at O and with the wheel Z at D. Since the wheel X is fixed, the point O will be the instantaneous center of the wheel Y. The point D is common to the wheel Z and the wheel Y, and Dd will be the absolute velocity of the point D. Draw Od, and erect a perpendicular to OD at E, cutting Od at e. Then the vector Ee represents the absolute velocity of the center of the wheel Y.

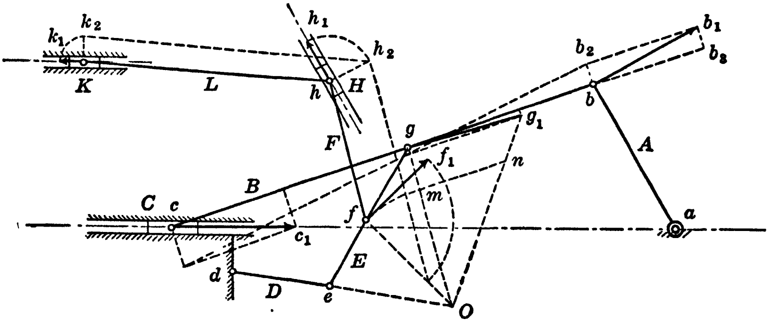


FIG. 81

85. Example 7.

GIVEN. The Joy reversing-valve linkage $ABCDEFHLK$ is shown in Fig. 81, where a and d are fixed points. The steam-engine crank-and-crosshead mechanism is represented by the links *A*, *B*, and *C*. The links *D*, *E*, *F*, *L*, and the slide *H* are added to the steam-engine crank and crosshead to operate the valve *K*. The vector cc_1 represents the absolute velocity of the engine piston.

REQUIRED. The absolute velocity of the valve *K*.

CONSTRUCTION. By composition and resolution of velocities, the vector gg_1 represents the absolute velocity of *g*, and bb_1 represents the absolute velocity of *b*. A line drawn through *g* perpendicular to gg_1 will intersect *de*, extended, at *O*. The point *O* is the instantaneous center of the link *E*. By the method of instantaneous centers, ff_1 is found to represent the absolute velocity of *f*. By the method of transference of velocities from *f*, the vector Kk_1 is found to represent the absolute velocity of *K*.

86. Example 8.

GIVEN. The linkage *ABCDE* is shown in Fig. 82. The point *a* is fixed. The points *b* and *e* are points on the disk *A* rotating about the fixed center *a*. The surface speed of the disk is represented by the vector *Xx*.

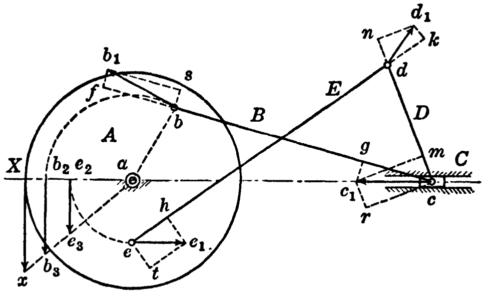


FIG. 82

REQUIRED. The absolute velocity of the point *d*.

CONSTRUCTION. Draw *aX* and *ax*. Rotate *ab* to ab_2, and draw b_2b_3 perpendicular to *aX*, intersecting *ax* at b_3. Draw bb_1 perpendicular to *ab* and equal to b_2b_3. The vector bb_1 represents the absolute velocity of *b*. Similarly, the vector ee_1 represents the absolute velocity of *e*.

Resolve bb_1 into components *bf* and *bs* along and perpendicular to *cb*. Draw *cg* equal to *bf*, and draw gc_1 perpendicular to *cb* intersecting the path of *c* at c_1. The vector cc_1 represents the absolute velocity of *c*. Resolve cc_1 into components *cm* and *cr*, along and perpendicular to *cd*. Draw *dn* equal to *cm*, the velocity of *c* along the link *D*. The vector *dn* represents a component of the velocity of *d* due to the translation of *D* along its center line *cd*. This component, added to an unknown component perpendicular to *cd* due to rotation of *D* about *c*, will give the absolute velocity of *d*. Draw nd_1 of unknown length perpendicular to *dn* at *n* to represent this unknown component.

Similarly, resolve ee_1 into components *eh* and *et*, and draw *dk* equal to *eh*. The vector *dk* represents a component of the velocity of *d* due to the translation of the link *E* along its center line *ed*. This component, added to an unknown component due to rotation of *E* about *e*, will give the absolute velocity of *d*. Draw kd_1 perpendicular to *dk* at *k* to represent this unknown component. Draw dd_1 through the intersection of nd_1 and kd_1. The vector dd_1 represents the absolute velocity of *d* in magnitude and direction.

87. Velocity Vector Images. In the four-bar linkage shown in Fig. 83, the velocities of *B* and *C* are of course perpendicular to

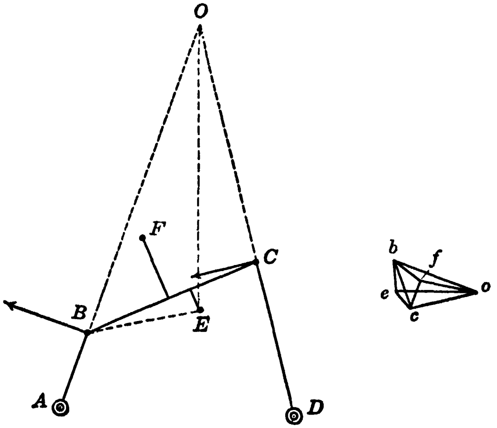


Fig. 83

the cranks *AB* and *DC*, and the instantaneous center of the link *BC* is at *O*. In the vector diagram shown at the right of the figure, *ob* represents the absolute velocity of *B*, and *oc* represents the absolute velocity of *C*. By vector subtraction *cb* is found to represent the velocity of *B* relative to *C* or v_{BC}. Since *O* is the instantaneous center, we know that

$$\frac{v_B}{v_C} = \frac{OB}{OC} = \frac{ob}{oc}.$$

Since *ob* and *oc* are perpendicular to *OB* and *OC* respectively, it follows that the triangles *obc* and *OBC* are similar. Then *cb* is perpendicular to *CB*. This indicates that the relative velocity of two points on the body is perpendicular to the line joining the two points, which is in agreement with the ideas developed in § 38.

The velocity of the point *E* is perpendicular to *OE*. In the vector diagram draw *oe* parallel to *OE* and equal to the velocity of *E*. Since *oe* and *ob* are parallel to v_E and v_B respectively, and proportional to *OE* and *OB*, we find that the triangles *obe* and *OBE* are similar, that *be* is perpendicular to *BE*, and that the vector *be* represents the velocity of *E* relative to *B*. Similarly *eb* represents the velocity of *B* relative to *E*. The same line of reasoning shows that the vector *ec* represents the velocity of *C* relative to *E*. Note that the triangles *bec* and *BEC* are similar, their sides being mutually perpendicular, and that in the vector diagram the vectors representing relative velocities are in every case perpendicular to the line joining the two points in the body whose relative velocities are being considered.

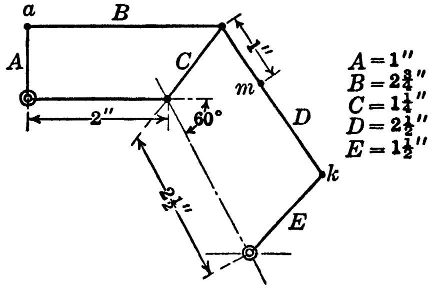
The vector diagram is the ***vector image*** of the body *BC*, and may be used to find the relative velocities of any two points, or the absolute velocity of any point on the body if the absolute velocities of two points are known. For example, in Fig. 83 the absolute velocity of the point *F* may be found as follows: Since v_B and v_C are known, the vectors *ob* and *oc* can be drawn. Draw *bf* perpendicular to *BF*, and *cf* perpendicular to *CF*. The vectors *bf* and *cf* represent the relative velocities v_{FB} and v_{FC}. A line drawn from *o* to *f* will represent the velocity v_{FO}, and since *O* is stationary (instantaneous center), this is the absolute velocity of *F* or v_F.

If, in Fig. 83, the link AB is held stationary the point O will be the centro or common velocity point of the links BC and AD. The vectors bc, be, and bf will then represent the absolute velocities v_C, v_E, and v_F. The vectors ob, oc, oe, and of will represent the relative velocities v_{BO}, v_{CO}, v_{EO}, and v_{FO}.

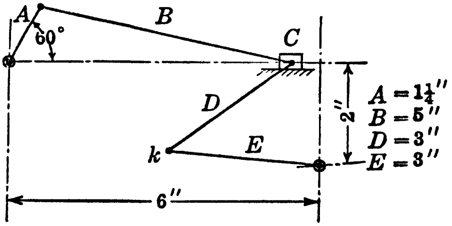
This method of determining velocities may be very convenient for the designer having a large amount of velocity analysis to do.

PROBLEMS

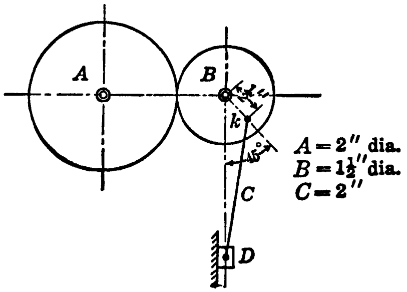
1–5. Locate all centros in the linkages shown in Problems 1 to 5



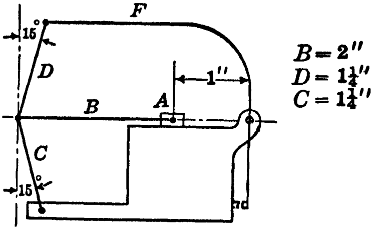
PROBLEM 1



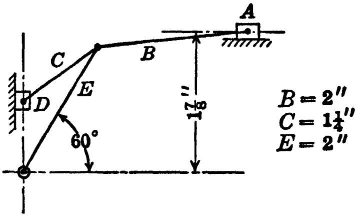
PROBLEM 2



PROBLEM 3



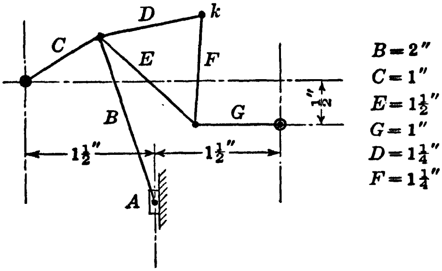
PROBLEM 4



PROBLEM 5

6. In Problem 1, assume the velocity of *a* to be one in. per sec. Determine the velocity of *k* and *m* by the method of centros.

7. In Problem 2, *A* rotates at 3.186 revolutions per sec. Determine the absolute velocity of *k* and the velocity of *k* relative to *C* by the method of centros.



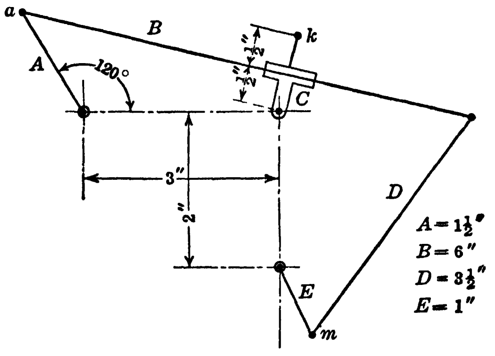
PROBLEM 10

8. In Problem 3, the wheel *A* rotates at 10 r.p.m. Determine the velocity of *k* and *D* by the method of centros.

9. In Problem 5, *A* moves 1 ft. per sec. Determine the velocity of *D* by the method of centros.

10. In the figure, the velocity of *A* is 15 ft. per sec. Determine the velocity of *k* by the method of centros.

11. In the figure, the velocity of *a* is represented by a vector 1 in. long. Determine the velocity of *k* by the centro method.

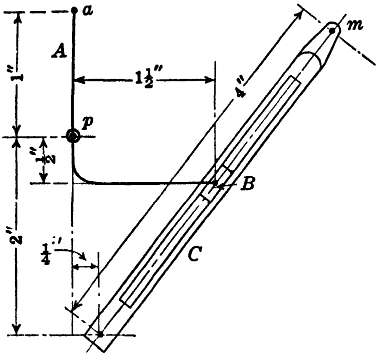


PROBLEM 11

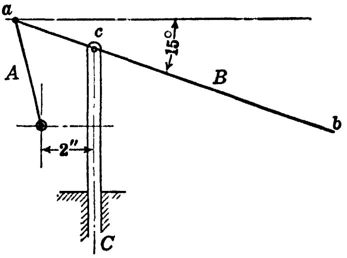
12. In the figure for Problem 11, assume the velocity of the middle point of *A* to be represented by a vector ¾ in. long. Determine the velocity of *m* by the method of centros.

13. A shaper mechanism is similar to Fig. 45 except that the link *D* is omitted, and the oscillating link *C* is fitted with a slot working on the pin *e* of the sliding block *E*. The center line of the path of *e* passes 3 in. above *a*. $A = 2\frac{1}{2}$ in., $ac = 4$ in. and *A* rotates at 10 r.p.m. Determine the stroke of *E* and the velocity of *E* when *A* makes an angle of 45° with the vertical.

14. In the figure the arm *A* is vertical at the instant considered and point *a* has a velocity represented by a vector ¾ in. long. Determine the velocity of the point *m*.



PROBLEM 14



PROBLEM 15

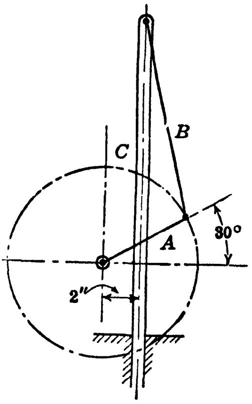
15. In the cistern pump shown, determine the velocity of the pump-rod *C* if the velocity of point *b* is represented by a vector 1½ in. long.

$$A = 4 \text{ in.},$$
$$B = 15 \text{ in.},$$
$$ac = 3 \text{ in.}$$

16. If the crank *A* rotates at 25 r.p.m., determine the velocity of the wind-mill pump-rod *C*.

$$A = 6 \text{ in.},$$
$$B = 12 \text{ in.}$$

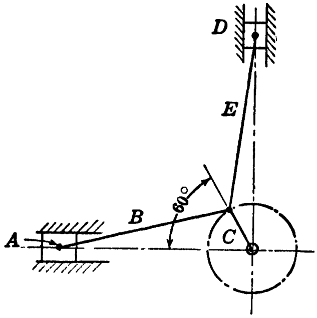
17. Given the velocity of piston *A*, equal to 200 ft. per min. Determine the ve-



PROBLEM 16

locity of the piston *D* and the revolutions per min. of the crank *C*.

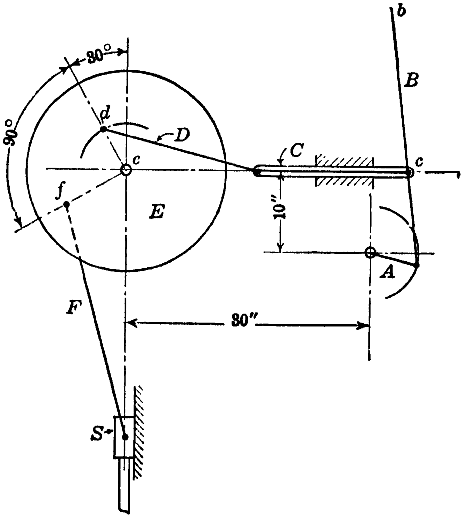
$C = 2$ ft., $B = 8$ ft. $= E$.



PROBLEM 17

18 In the pumping device shown, determine the velocity of the slide *S* if the velocity of *b* is a vector $1\frac{1}{2}$ in. long.

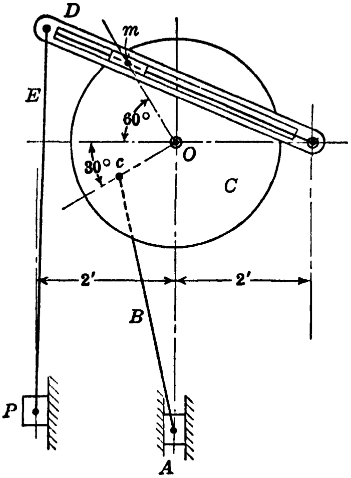
$A = 6$ in.,
$B = 36$ in.,
$C = 18$ in.,
$D = 20$ in.,
$F = 30$ in.,
$bc = 24$ in.,
$ed = 6$ in.,
$ef = 9$ in.



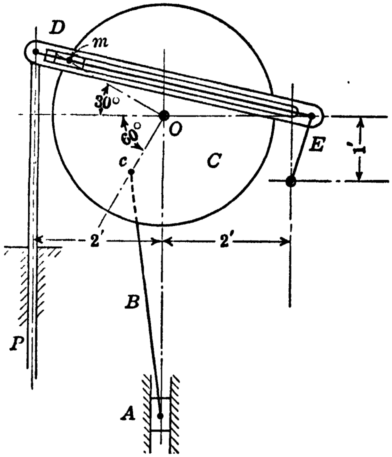
PROBLEM 18

19. In the pumping jack illustrated, find the velocity of *P* if the velocity of the piston *A* is 300 ft. per min.

B = 4 ft.,
D = 4 ft. 3 in.,
E = 6 ft.,
om = 1½ ft.,
oc = 1 ft.



PROBLEM 19



PROBLEM 20

20. In the pumping jack illustrated, find the velocity of *P* if the velocity of *A* is 250 ft. per min.

B = 4 ft.,
D = 4 ft. 3 in.,
E = 1 ft.,
om = 1½ ft.,
oc = 1 ft.

21. Assuming that the force exerted is inversely proportional to the velocity when friction is disregarded, find the force exerted on the rivet die *D*, in the figure of Problem 5, if the pressure on the piston *A* is 2000 lbs. Determine this force when *E* makes an angle of 75° with the horizontal.

CHAPTER VIII

ACCELERATION AND ACCELERATION DIAGRAMS

88. Acceleration and Inertia Forces. When the velocity of a body is changing, the body is undergoing either positive or negative acceleration. The latter is often called deceleration. According to Newton's second law, if there is an acceleration of either positive or negative character, there must be an external force acting on the body to produce this acceleration. This implies an equal and opposite force arising in the body resisting the acceleration. This resisting force is called an ***inertia force.*** In machinery having parts moving at high speeds, these inertia forces are large; and in the design of the machine parts, these forces must be known, in order that the machine parts may be made sufficiently large to withstand them. These forces are equal to the acceleration times the mass of the body. Therefore if the acceleration can be determined, the force can be determined.

89. Acceleration from Velocity-Time Diagram. In general, the acceleration can be determined from the velocity-time diagram. From the definition of acceleration, it is known that

$$a = \frac{dv}{dt},$$

where a = acceleration in ft. per second per second, dv = change of velocity, in ft. per second, occurring in an infinitesimal interval of time and dt = the infinitesimal interval of time, in seconds.

Figure 84 shows the velocity-time diagram for a body moving along the line OX. The distance of any point on the curve from OX, measured perpendicular to OX, represents the magnitude of the velocity, or speed. The direction of the velocity is along the line OX. The slope of the velocity-time curve at any instant of time represents the acceleration at that instant, tangent to the path. This is evident since the slope of the tangent to the velocity curve is

$$\text{slope} = \tan\theta = \frac{dv}{dt} = a.$$

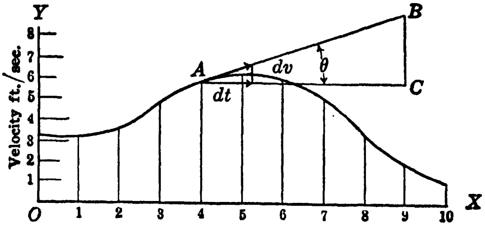


FIG. 84

To determine the actual magnitude of the acceleration, the slope must be interpreted by the proper scale. Draw AB tangent to the velocity curve, AC parallel to OX and of a length representing a known number of seconds, say 5 seconds. Draw CB perpendicular to OX. To scale, CB represents the increase in velocity if the acceleration remained constant 5 seconds. The line CB scales 3½ feet per second. Then the acceleration at A is

$$a = \frac{dv}{dt} = \frac{CB}{AC} = \frac{3.5}{5} = 0.7 \text{ ft. per sec. per sec.}$$

If AC is made equal to the length representing 1 second, then CB will be the acceleration in feet per second per second, to the same scale as the velocity curve, and may be plotted directly as an ordinate of the acceleration-time curve.

90. Acceleration from Velocity-Space Diagrams. The acceleration can also be found from the velocity-space diagram by means of the following rule: *The acceleration varies as the subnormal to the velocity-space diagram.* A very convenient method of plotting the acceleration-space curve from the velocity-space curve is as follows.

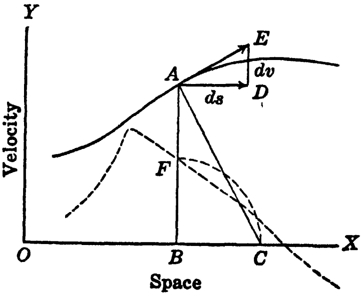


FIG. 85

In Fig. 85, at any point on the velocity-space curve, draw AC normal to the curve, and

AB perpendicular to the base line, OX. Draw AE tangent to the curve, AD parallel to OX, and DE perpendicular to OX. Plot BF equal to BC. The point F is a point on the acceleration-space diagram to *a scale which can be found* as follows. Let S_s = space scale, S_v = velocity scale, and S_a = acceleration scale; then we have

$$a = \frac{dv}{dt} = \frac{dv}{dt} \times \frac{ds}{ds} = \frac{dv}{ds} \times \frac{ds}{dt},$$

$$v = \frac{ds}{dt} = AB \times S_v, \qquad \frac{dv}{ds} = \frac{ED \times S_v}{AD \times S_s}.$$

Hence

$$a = \frac{ED \times S_v}{AD \times S_s} \times AB \times S_v = \frac{ED \times AB}{AD} \times \frac{S_v^2}{S_s}.$$

From the figure, we see that

$$\frac{BC}{AB} = \frac{ED}{AD}, \quad \text{or} \quad BC = \frac{ED \times AB}{AD}.$$

Hence

$$a = BC \times \frac{S_v^2}{S_s}.$$

Since $a = BC \times S_a$, we have

$$S_a = \frac{S_v^2}{S_s}. \tag{28}$$

91. Mathematical Determination of Acceleration. Since acceleration is the time rate of change of velocity, and velocity is the time rate of movement through space, we can determine the acceleration by differentiation of the velocity-time equation or by double differentiation of the space-time equation, if either of these equations is known. Several linkages of special interest will be discussed to illustrate the method of procedure.

92. Scotch Yoke. In Fig. 86, the crank AB rotates with uniform angular velocity, the pin B sliding in the groove EF imparting a reciprocating motion to the rod CD. The displacement of the rod CD is represented by the displacement of the point C

on the center line of the slot EF. The displacement of C measured from its extreme position to the right is

$$\text{displacement} = s = R - R\cos\theta = R(1 - \cos\theta). \tag{29}$$

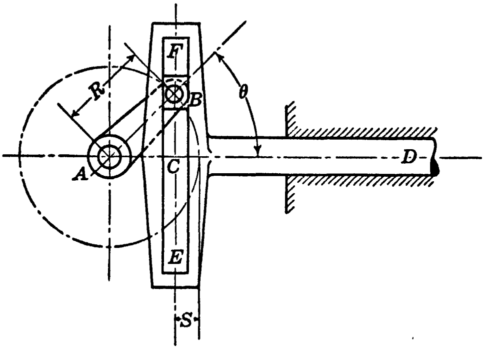


FIG. 86

The velocity of C is the derivative of the displacement with respect to time. Thus we have

$$\text{velocity} = \frac{ds}{dt} = R\sin\theta\frac{d\theta}{dt} = \omega R\sin\theta, \tag{30}$$

where ω = angular velocity in radians per second.

The acceleration of C is the derivative of the velocity with respect to time. Thus we have

$$\text{acceleration} = \frac{dv}{dt} = R\omega\cos\theta\frac{d\theta}{dt} = R\omega^2\cos\theta. \tag{31}$$

Note that the crank AB gives the sliding rod CD simple harmonic motion.

93. Crank and Crosshead Mechanism. Accelerations in this linkage are of special interest on account of the large number of high-speed steam and gas engines now being built, and the modern trend toward still higher speeds.

In the engine, the piston drives the linkage through the crosshead, causing the crank to revolve. The weight and inertia of the flywheel and other revolving parts keep the crank revolving at very nearly constant angular velocity. The velocity of the crosshead is always varying; during part of the stroke the crosshead is accelerating, and during the remainder of the stroke it is decelerating.

In Fig. 87, the crank AB is the driver and rotates with uniform velocity. From the geometry of the figure, the displacement, s, of the crosshead C is

$$\text{(32)}\quad \text{displacement} = s = (R + L) - (R\cos\theta + L\cos\phi),$$

but $R\sin\theta = L\sin\phi$, from which

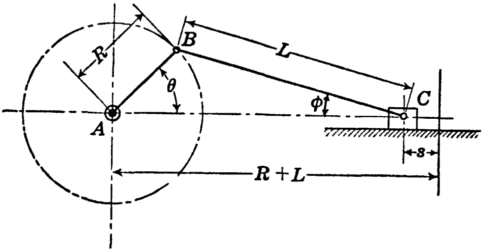


FIG. 87

$$\cos\phi = \left\{1 - \left(\frac{R}{L}\right)^2 \sin^2\theta\right\}^{1/2},$$

and

$$\text{(33)}\quad s = R(1 - \cos\theta) + L\left[1 - \left\{1 - \left(\frac{R}{L}\right)^2 \sin^2\theta\right\}^{1/2}\right].$$

The velocity of the crosshead C is

$$v = \frac{ds}{dt} = R\sin\theta\frac{d\theta}{dt} + L\left[\frac{\left(\frac{R}{L}\right)^2 \sin\theta\cos\theta}{\left\{1 - \left(\frac{R}{L}\right)^2 \sin^2\theta\right\}^{1/2}}\frac{d\theta}{dt}\right],$$

$$\text{(34)}\quad v = R\omega\sin\theta + L\frac{\left(\frac{R}{L}\right)^2 \omega\sin 2\theta}{2\left\{1 - \left(\frac{R}{L}\right)^2 \sin^2\theta\right\}^{1/2}},$$

$$\text{(35)}\quad v = R\omega\left[\sin\theta + \frac{R}{L}\frac{\sin 2\theta}{2\left\{1 - \left(\frac{R}{L}\right)^2 \sin^2\theta\right\}^{1/2}}\right],$$

or, approximately, with the ordinary ratios of R/L from $\frac{1}{3}$ to $\frac{1}{6}$,

$$(36) \qquad v = R\omega\left(\sin\theta + \frac{R}{L}\frac{\sin 2\theta}{2}\right).$$

The acceleration of the crosshead C is

$$(37) \quad a = \frac{dv}{dt} = R\omega^2\left[\cos\theta + \frac{\frac{R}{L}\cos 2\theta + \left(\frac{R}{L}\right)^3\sin^4\theta}{\left[1 - \left(\frac{R}{L}\right)^2\sin^2\theta\right]^{3/2}}\right],$$

or, approximately,

$$(38) \qquad a = R\omega^2\left(\cos\theta + \frac{R}{L}\cos 2\theta\right).$$

This method of double differentiation of the space-time equation or single differentiation of the velocity-time equation may be applied to most mechanisms, but in general will give rather cumbersome equations. In most cases, the vector analysis of acceleration is more convenient and is sufficiently accurate.

94. Tangential and Normal Acceleration. In Fig. 88a, assume a particle p to move along the path mp_1p_2n in such a way that when it is at p_1, its velocity is represented by the vector p_1a or v_1; and when it is at p_2, its velocity vector is p_2b, or v_2. In this case, v_2 is greater than v_1 and also acts in a different direction. In Fig. 88b, the vector OA is equal and parallel to p_1a and represents v_1. Also OB represents v_2. It is seen that OB is the vector sum of OA and AB, and that AB must be added to v_1 to obtain v_2. Therefore AB is the change in velocity, Δv, during the time required for the particle to move from p_1 to p_2. Then the average acceleration over this time interval is

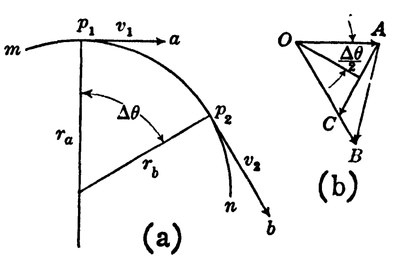


FIG. 88

$$\text{average } a = \frac{\Delta v}{\Delta t} = \frac{AB}{\Delta t}$$

in which Δt is the time measured in seconds during which the particle has moved from p_1 to p_2. As the point p_2 is taken closer and closer to p_1, the time interval, Δt, approaches zero, and the average acceleration becomes the instantaneous acceleration:

$$a = \frac{dv}{dt} = \text{instantaneous acceleration.}$$

In Fig. 88b, OC has been laid off equal to OA or v_1. The change of velocity AB or Δv is the vector sum of two vectors, AC and CB. The change in magnitude of the velocity is represented by CB, and AC represents the velocity change due to change of direction only. It follows that

$$\Delta v = \text{vector sum } (AC + CB).$$

Now

$$AC = 2\,OA \sin\frac{\Delta\theta}{2} = 2v_1 \sin\frac{\Delta\theta}{2},$$

$$CB = OB - OC = v_2 - v_1,$$

$$\text{average } a = \frac{\Delta v}{\Delta t} = \text{vector sum}\left[\frac{2v_1 \sin\frac{\Delta\theta}{2}}{\Delta t} + \frac{(v_2 - v_1)}{\Delta t}\right].$$

As Δt approaches zero, $v_2 - v_1$ approaches dv, the angle $\Delta\theta$ approaches zero and $\sin(\Delta\theta/2)$ approaches the angle $d\theta/2$. Then the instantaneous acceleration is

$$\begin{aligned} a &= \text{vector sum}\left(v_1\frac{d\theta}{dt} + \frac{dv_1}{dt}\right) \\ a &= \text{vector sum } (v\omega + r\alpha) = \text{vector sum } (r\omega^2 + r\alpha), \end{aligned} \tag{39}$$

where v = instantaneous velocity, α = angular acceleration, ω = angular velocity, and r = radius of curvature of the path.

The direction of these two component accelerations must be carefully noted. In Fig. 88b, the vector AC is perpendicular to the bisector of $\Delta\theta$, and as $\Delta\theta$ becomes $d\theta$, AC becomes perpendicular to OA. Therefore the component acceleration $v\,d\theta/dt$ is, at the instant under consideration, normal to the path of the particle or along the radius of curvature and always toward the center of curvature. This component is called the ***normal ac-***

celeration. Also dv/dt is tangent to the path, and this component is called the ***tangential acceleration.*** Then

$$(40) \qquad a = \text{vector sum } (a_n + a_t),$$

$$(41) \qquad a = \sqrt{a_n^2 + a_t^2},$$

where

$$a_n = v\omega = \frac{v^2}{r} = r\omega^2 = \text{normal acceleration},$$

$$a_t = \frac{dv}{dt} = r\alpha = \text{tangential acceleration}.$$

The instantaneous acceleration makes an angle, ϕ, with the radius of curvature such that

$$(42) \qquad \tan\phi = \frac{a_t}{a_n} = \frac{r\alpha}{r\omega^2} = \frac{\alpha}{\omega^2}.$$

95. Acceleration of Particles in a Rigid Body. It is shown in § 38 that the velocity of any particle in a rigid body is the resultant of the velocity of some point in the body and a velocity relative to this point. Similarly, *the acceleration of any particle of the body is the resultant of the acceleration of some point in the body and the acceleration of the particle relative to this point,* that is, the acceleration due to rotation of the body about the point. The absolute acceleration of particle B, in Fig. 89, is

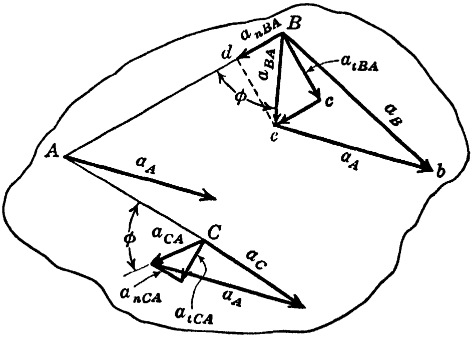


FIG. 89

$$a_B = \text{vector sum } (a_A + a_{BA}) = \text{vector sum } (a_A + a_{nBA} + a_{tBA}),$$

where a_A = acceleration of A, a_{nBA} = normal acceleration of B relative to A, and a_{tBA} = tangential acceleration of B relative to A.

Note that on the rigid body the angular velocity ω and the angular acceleration α are the same for all lines such as AB, AC, etc. Therefore the angle ϕ, which the acceleration of any particle relative to A makes with the radius from the particle to A, is the same for all particles of the body.

It follows from what precedes that, if the absolute acceleration of any particle in a rigid body is known, together with the angular velocity and angular acceleration of the body, the absolute acceleration of all particles in the body is completely determined.

96. Transferring of Acceleration. In rigid bodies forming the links of a mechanism, it is seldom that the angular velocity, or angular acceleration, or their ratio α/ω^2 is known. Usually, however, enough is known about the driving link (generally a crank) so that the acceleration of one particle can be calculated and represented by a vector on the drawing of the linkage. The accelerations of other particles can then be found, as shown in the following articles.

97. Acceleration Scales. The work and computations required in graphical work with acceleration vectors are materially decreased if definite relations between the space, velocity, and acceleration scales are adopted.

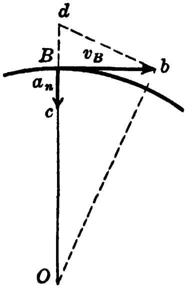


FIG. 90

In Fig. 90, let OB represent to scale the radius of curvature of the path of B, Bb the velocity of B, and Bc the normal acceleration of B. Then

$$r = OB \times S_s, \quad v = Bb \times S_v, \quad a_n = Bc \times S_a,$$

where r is measured in feet, v in feet per second, a in feet per second per second, and S_s, S_v, S_a are the space, velocity, and acceleration scales used in the drawing. Now we have

$$a_n = \frac{v^2}{r} = \frac{(Bb \times S_v)^2}{OB \times S_s} = \frac{(Bb)^2}{OB} \times \frac{S_v^2}{S_s},$$

$$a_n = Bc \times S_a.$$

If the scales are chosen so that

$$S_a = \frac{S_v^2}{S_s},$$

we have

$$Bc = \frac{(Bb)^2}{OB}, \qquad \frac{Bc}{Bb} = \frac{Bb}{OB}.$$

The length Bb on the drawing is then a mean proportional to OB and Bc. It follows that *if the vector which represents the normal acceleration, namely Bc, is made equal in length to $(Bb)^2/OB$, as measured in inches on the drawing, the acceleration scale is determined by*

$$S_a = \frac{S_v^2}{S_s}.$$

This can be done graphically by drawing a line bd perpendicular to Ob and making Bc equal to Bd as in Fig. 90.

For example, assume the radius of curvature of the path to be 10 feet and the space scale 1 inch equal to 2 feet on a drawing similar to Fig. 90. Assume the velocity to be 50 feet per second, and the velocity scale 1 inch equal to 40 feet per second. On the drawing, the radius of curvature OB will be 5 inches, the velocity vector Bb will be 1.25 inches, and Bd will be 0.3125 inches. Then the normal acceleration should be represented by a vector Bc equal to Bd or 0.3125 inches long. The acceleration scale will be

$$S_a = \frac{S_v^2}{S_s} = \frac{40^2}{2} = 800,$$

or 1 inch = 800 feet per second². Then we may write

$$a_n = 800 \times 0.3125 = 250 \text{ ft. per second}^2.$$

98. Acceleration in the Four-Bar Linkage. The applications of the principles developed in the preceding articles will be applied to several linkages. Consider first the four-bar linkage in Fig. 91. Let Bb represent, to scale, S_v, the velocity of B. Extend AB, and draw bf perpendicular to Ab, the point A being the center of curvature of the path of B. Then Bf is the magnitude of the normal acceleration of B to the scale $S_a = S_v^2/S_s$, but opposite in direction. Lay off Bg equal to Bf as the vector representing the normal acceleration of B. If AB has an angular

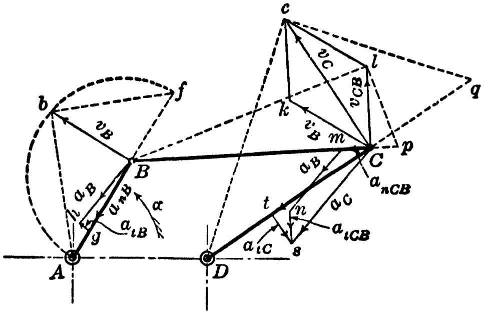


FIG. 91

acceleration α, the tangential acceleration of B must be calculated and added vectorially to Bg, in order to give Bh the absolute acceleration of B.

Since this is an acceleration problem, it is more convenient to calculate the normal and tangential accelerations and to make Bf equal to $-a_n$. If this is done, Bb, the velocity vector for B, is found by drawing a semi-circle on Af as a diameter and drawing Bb perpendicular to AB and intersecting the semi-circle at b.

The acceleration of C is known to be

$$a_C = \text{vector sum } (a_B + a_{CB}) = \text{vector sum } (a_B + a_{nCB} + a_{tCB}).$$

The acceleration of B is known, but the normal and tangential accelerations of C relative to B depend on the angular velocity and acceleration of the link BC. To find the angular velocity of C, draw Ck equal and parallel to Bb, Cl and kc perpendicular to BC, and Cc perpendicular to DC. The vector Cl is the velocity of C relative to B. Draw Bl, and lp (perpendicular to Bl) intersecting BC extended at p. Make Cm equal to Cp. Then Cm represents the normal acceleration of C relative to B.

The tangential acceleration of C relative to B cannot be found directly, since the angular acceleration is unknown. Consider, however, the particle C as being on the crank DC, Cc representing the absolute velocity of C. Draw Dc and cq perpendicular to Dc and intersecting DC extended at q. Draw Ct equal and opposite to Cq. The vector Ct represents the normal acceleration of C relative to D, and since D is a fixed point, Ct represents the absolute normal acceleration of C.

The vector Ct added to the absolute tangential acceleration of C must give the absolute acceleration of C. The direction of the tangential acceleration is perpendicular to Ct. Draw ts to represent the direction of this acceleration of unknown magnitude.

Now draw Cm, the normal acceleration of C relative to B. Add to this mn, equal and parallel to Bh, the absolute acceleration of B. Draw ns perpendicular to BC to represent the direction of the unknown tangential acceleration of C relative to B. The point s at the intersection of ts and ns is the terminus of the vector Cs, which represents the absolute acceleration of C.

99. Acceleration of Particles Not at the Connecting Pins. It is now possible to determine the acceleration of any point on the

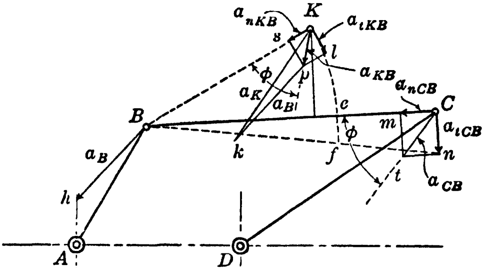


FIG. 92

link BC. In Fig. 92, consider that the velocities and accelerations of B and C are those just determined.

Since the tangential acceleration is a_t equal to $r\alpha$, the tangential acceleration of K relative to B may be found from the relation

$$a_{tKB} = a_{tCB} \times \frac{BK}{BC}.$$

Graphically, this is found as follows. The acceleration of C relative to B is represented by the vectors Cn and Cm equal respectively to the vectors *ns* and *Cm* found in Fig. 91. Draw *Bn*, make *Be* equal to *BK*, draw *ef* perpendicular to *BC*, intersecting *Bn* at *f*. Draw *Kl* perpendicular to *BK* and equal to *ef*. Then *Kl* represents the tangential acceleration of *K* relative to *B*.

The direction of the acceleration of *K* relative to *B* is found by drawing *Kp* making the angle ϕ with *BK*, where

$$\tan \phi = \frac{Cn}{Cm},$$

since the angle between the total acceleration and the normal acceleration is the same for all particles of a rotating body.

Draw *lp* parallel to *BK* intersecting *Kp* at *p*. Then *Kp* represents the acceleration of *K* relative to *B*. When to this vector is added a vector *pk*, equal to and parallel to *Bh* which is the absolute acceleration of *B*, the vector *Kk* is obtained which represents the absolute acceleration of *K*.

100. Crank and Crosshead. The acceleration of the crosshead may be determined by the method used on the four-bar linkage, remembering that the directions of the absolute acceleration of the crosshead are always along the same straight line, that is, along the straight path of the crosshead. On account of the importance of this mechanism, special graphical methods of construction have been developed, two of which are given here.

101. Klein's Construction. In Fig. 93, draw *AE* perpendicular to *CA*, the path of the crosshead. With *CB* as a diameter, draw the arc *CGBH*. Extend *CB* to intersect *AE* at *F*. With *BF* as a radius, draw the arc *GFH* cutting the arc *CGBH* at *G* and *H*. Draw the line *GH* to intersect *CA* at *J*.

If the length of the crank *AB* represents to scale the acceleration of the crank pin *B*, then *JA* represents to the same scale the acceleration of the crosshead *C*. The acceleration of *B* is

$$a_B = \omega^2 BA = \frac{\pi^2 N^2 \times BA}{900}.$$

When *J* is between *A* and *C*, the acceleration is positive; and when *J* is to the right of *A*, the acceleration is negative. The direction and sense of the acceleration is always from *J* toward *A*.

Draw CK perpendicular to CA and equal to JA. Repeat for any number of positions of C, and draw a smooth curve through all the points corresponding to K. This is the acceleration-space diagram for the crosshead C.

102. Proof of Klein's Construction.[*] Draw Cc, in Fig. 93, equal to JA to represent the acceleration of C to some scale to be determined. Draw cm perpendicular to CB. Then Cm is the acceleration of C in the direction CB, which is made up of a normal acceleration due to rotation of C about B and an acceleration of the whole rod in the direction CB. Draw AM perpendicular to CBF and SN parallel to CA.

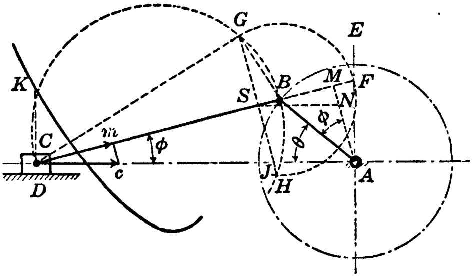


FIG. 93

From the construction

$$SN = JA = Cc, \quad \text{and} \quad Cm = SM = SB + BM.$$

If it can be shown that SB represents to scale the normal acceleration of C due to rotation about B, and that BM is the acceleration of B in the direction CB, Klein's construction will be proved.

Consider the acceleration of B, which for uniform rotation is in the direction BA and is

$$a_B = \frac{v_B^2}{r} = \frac{AB^2 \times S_v^2}{AB \times S_s} = AB\frac{S_v^2}{S_s} = AB \times S_a,$$

where

$AB \times S_v =$ velocity of B (§ 54),

$AB \times S_a =$ acceleration of B,

$S_v =$ scale of velocity,

$S_s =$ scale of space, or the drawing scale,

$S_a =$ scale of acceleration $= \dfrac{S_v^2}{S_s}$.

---

[*] This construction is a variation of the proof given by W. E. Dalby in *Balancing of Engines*, § 105.

Since AM is perpendicular to CBF, and BA is the acceleration of B, it follows that BM is the acceleration of B in the direction CB.

In § 53, it is shown that AF represents the velocity of C to the same scale that AB represents the velocity of B. It follows that BF represents the magnitude of the velocity of C relative to B, or

$$v_{CB} = BF \times S_v = BG \times S_v. \tag{43}$$

Draw CG and GB. Since the triangles CSG and GSB are similar, we have

$$\frac{BS}{BG} = \frac{BG}{CB},$$

$$BS = \frac{BG^2}{CB} = \frac{v_{CB}^2}{S_v^2} \times \frac{1}{CB} = \frac{v_{CB}^2}{S_v^2} \times \frac{S_s}{r}, \tag{44}$$

$$BS = \frac{v_{CB}^2}{r} \times \frac{S_s}{S_v^2} = a_{nCB} \times \frac{1}{S_a}, \tag{45}$$

or the normal acceleration of C relative to B is

$$a_{nCB} = SB \times S_a. \tag{46}$$

The total acceleration of C in the direction CB is then

$$a_{nCB} + a_B \sin \phi = SB + BM = SM = CM, \tag{47}$$

and from the figure

$$Cc = JA = SN = \text{the acceleration of } C,$$

which was to be proved.

103. Bennett's Construction. In Fig. 94, locate the point M on the connecting-rod CB so that BA is the mean proportional between BM and BC; then we may write

$$BM \times BC = BA^2.$$

The point M can be located when the crank AB is at right angles to CA, if a line is drawn through A perpendicular to the connecting-rod. The intersection of the perpendicular with the connecting-rod is the desired point which is shown as M_1.

After M is located, draw ML perpendicular to CB, intersecting CA at L. Draw LK perpendicular to CA, intersecting CB at

K. Draw *KJ* perpendicular to *CB*, intersecting *CA* at *J*. The line *JA* represents the acceleration of the crosshead *C*, to the same scale that *AB* represents the acceleration of the crank pin *B*.

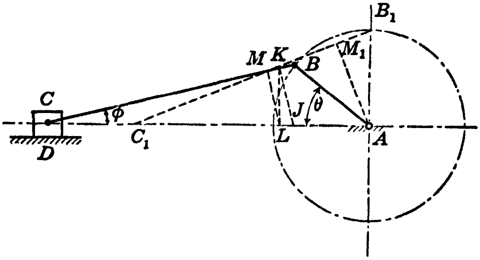


FIG. 94

104. Proof of Bennett's Construction.* In Fig. 94, let θ equal the crank angle *CAB* and ϕ equal the connecting-rod angle *BCA*. Then

$$(48) \qquad BC \sin \phi = AB \sin \theta, \quad \text{or} \quad l \sin \phi = c \sin \theta,$$

where l is the length of the connecting-rod *BC* and c is the length of the crank *AB*. Differentiating with respect to the time, we find

$$(49) \qquad l \cos \phi \frac{d\phi}{dt} = c \cos \theta \frac{d\theta}{dt} = c\omega \cos \theta.$$

Now the distance of the crosshead *C* from *A* is

$$(50) \qquad x = c \cos \theta + l \cos \phi.$$

Differentiating with respect to the time, we find

$$(51) \qquad \frac{dx}{dt} = -c \sin \theta \omega - l \sin \phi \frac{d\phi}{dt},$$

or, by (49),

$$(52) \qquad \frac{dx}{dt} = -(c \sin \theta + c \cos \theta \tan \phi)\omega.$$

Differentiating again with respect to the time, we find the acceleration of C to be

---

* G. T. Bennett of Emmanuel College, Cambridge, 1902. Proof follows that given by W. E. Dalby in *Balancing of Engines*, Third Edition, Appendix 1.

$$
\begin{aligned}
&a_C = \frac{d^2x}{dt^2} \\
(53)\qquad &= -\left(\cos\theta\cdot\omega + c\cos\theta\sec^2\phi\frac{d\phi}{dt} - c\omega\sin\theta\tan\phi\right)\omega,
\end{aligned}
$$

or, by (49), and by trigonometric reduction,

$$(54)\qquad a_C = -\,\omega^2\left(c\cos\theta + \frac{c^2\cos^2\theta}{l\cos^3\phi} - c\sin\theta\tan\phi\right),$$

or, by (48), and by trigonometric reduction,

$$
\begin{aligned}
(55)\qquad a_C &= -\,\omega^2\left(c\cos\theta + \frac{c^2 - l^2\sin^2\phi}{l\cos^3\phi} - \frac{l\sin^2\phi}{\cos\phi}\right), \\
&= -\,\omega^2\left(c\cos\theta + l\cos\phi - \frac{l^2 - c^2}{l}\sec^3\phi\right).
\end{aligned}
$$

From the figure, we have

$$AC = c\cos\theta + \cos\phi,$$

and

$$BC \times BM = c^2 = l(l - CM),$$

$$CM = \frac{l^2 - c^2}{l}, \qquad CL = \frac{l^2 - c^2}{l}\sec\phi,$$

$$CK = CL\sec\phi = \frac{l^2 - c^2}{l}\sec^2\phi, \quad CJ = CK\sec\phi = \frac{l^2 - c^2}{l}\sec^3\phi.$$

Equation (55) then becomes

$$
\begin{aligned}
(56)\qquad a_C &= -\,\omega^2(AC - AJ) \\
(57)\qquad &= -\,\omega^2 \times AJ \\
(58)\qquad &= -\frac{v_B{}^2}{c^2} \times AJ \\
(59)\qquad &= -\frac{a_B}{c} \times AJ = -\frac{a_B}{AB} \times AJ \\
(60)\qquad a_C &= -\,S_a \times AJ.
\end{aligned}
$$

It follows that JA represents the acceleration of the crosshead to the same scale that AB represents the acceleration of the crank pin B.

105. Acceleration of a Particle on the Connecting-Rod. When the acceleration of the cross-head, represented by the vector JA, Fig. 95, has been determined, the acceleration of any point on the connecting-rod, such as K, can be found as follows. The acceleration of B is along the radius BA, and is equal to $\omega^2 BA$. The acceleration of B relative to C is the vector difference of JA and BA. We then have

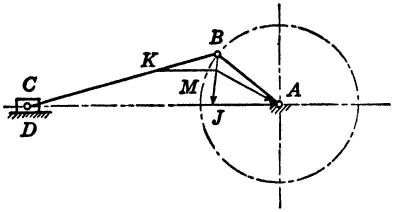


FIG. 95

$$\text{vector difference } (BA - JA) = BJ.$$

Therefore BJ represents the acceleration of B relative to C. The acceleration of K relative to C and the acceleration of B relative to C are proportional to the distances CK and CB. Draw KM parallel to CA, intersecting BJ at M. Then MJ represents the acceleration of K relative to C, since

$$\text{acceleration of } K \text{ relative to } C = BJ \times \frac{CK}{CB} = MJ.$$

The absolute acceleration of K is the vector sum of the acceleration of C and the acceleration of K relative to C. We then have

$$\text{vector sum } (JA + MJ) = MA.$$

Therefore the vector MA represents the acceleration of K to the same scale that BA represents the acceleration of B.

PROBLEMS

1. The equation of velocity for a certain body is $v = 12 + 6t + t^2$. Determine the equation for acceleration, and calculate the velocity and acceleration at the end of four units of time.

2. The equation of velocity for a certain body is $v = 20 - 4t + t^2$, where v is ft. per sec., and t is time in seconds.
   (a) Plot the velocity-time curve for 10 units of time.
   Scale 1 in. = 25 ft. per sec. and 1 in. = 2 seconds.
   (b) Determine graphically the acceleration when $t = 5$.

3. A body moves according to the equation $s = t^3 - 5t$ where s is in feet, and t is in seconds.

(a) Plot the space-time curve for 5 sec., plotting points for each ½ sec. Scale 1 in. = 50 ft.; 1 in. = 1 sec.

(b) Determine graphically from this curve the acceleration when $t = 3$. Give the numerical value of the acceleration in ft. per sec. per sec.

4. The plunger of a pump has a stroke of 12 in. and is driven from a crank by a Scotch yoke similar to Fig. 86. The crank makes 40 r.p.m.

(a) Determine the velocity of the plunger when the crank is at 0°, 45°, and 90° with the center line of the pump.

(b) Determine the acceleration of the plunger for the same crank positions.

5. A crank is rotating at the rate of 100 r.p.m. and at a particular instant the rotation is increasing at the rate of 50 r.p.m. per sec. Determine the tangential acceleration, the normal acceleration, and the absolute acceleration of a point 18 in. from the center of rotation.

6. The axis of a 3-ft. wheel is being accelerated to the right along a horizontal line at the rate of 20 ft. per sec. per sec. The wheel is revolving clockwise on its axis at the rate of 50 r.p.m. Determine the absolute acceleration of a particle on the rim of the wheel and on a radial line through the axis at 45° with the horizontal.

7. Determine graphically the acceleration of a particle on the rim of a 6-ft. flywheel when the flywheel rotates at 50 r.p.m. Give the numerical value of the acceleration in feet per sec. per sec. Scale of drawing 1 in. = 1 ft. Scale of velocity 1 in. = 10 ft. per sec.

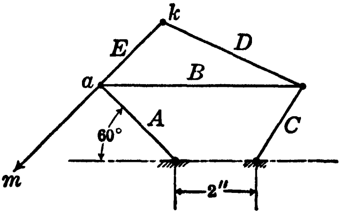
8. Two cranks, each 2 in. long, rotate on fixed centers 4 in. apart and are joined by a 4-in. connecting link which crosses the line of centers. The driving crank rotates at 10 r.p.m. When the driving crank makes an angle of 60° with the line of centers, determine the absolute acceleration of the end of the driven crank and of the middle of the connecting link.

9. The end of a rotating crank 2 ft. long has an acceleration of 150 ft. per sec. per sec. at an angle of 60° with the crank.

(a) Determine the tangential and normal accelerations.

(b) Determine the speed of rotation of the crank in r.p.m.

(c) Determine the angular acceleration.

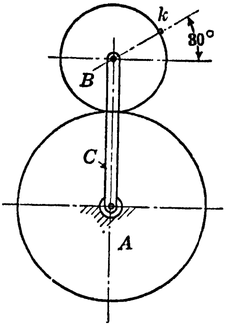


PROBLEM 10

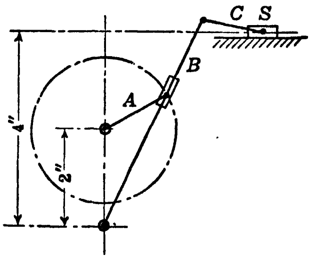
10. In the figure, the crank A rotates at 10 r.p.m. Determine the acceleration of the point k.

$A = \frac{3}{4}$ in., $B = 2$ in., $C = 1\frac{1}{4}$ in., $E = 1$ in., $D = 1\frac{1}{4}$ in.

11. In the figure, the wheel *A* is stationary, and the wheel *B* rolls on it without slipping. The diameter of *A* is 4 in., and the diameter of *B* is 2 in. If the arm *C* revolves 10 revolutions per min., determine the acceleration of the point *k*.



PROBLEM 11

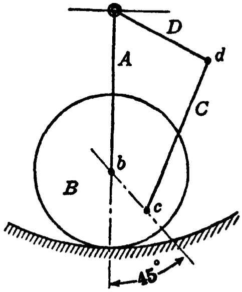


PROBLEM 12

12. In the figure, the crank *A* revolves uniformly at 10 r.p.m. Determine the acceleration of the slide *S* when *A* makes an angle of 30° with the horizontal. $A = 1\frac{1}{2}$ in., $B = 4\frac{3}{4}$ in., $C = 1\frac{1}{4}$ in.

13. In the figure, the arm *A* is revolving at the rate of 5 radians per sec. Determine the acceleration of the points *b*, *c*, and *d*.

$A = 1\frac{1}{2}$ in.,
$B = 1\frac{1}{2}$ in. diam.,
$C = 1\frac{1}{2}$ in.,
$D = 1$ in.,
$bc = \frac{1}{2}$ in.



PROBLEM 13

14. A gas engine has a 4-in. stroke and rotates at 800 r.p.m. The connecting-rod is 8 in. long.

(a) Determine mathematically the velocity of the piston when the crank makes angles of 0°, 45°, and 90° with the center line of the cylinder.

(b) Determine mathematically the accelerations of the piston for the same crank positions.

15. A steam engine has a stroke of 8 in., and the connecting rod is 18 in. long. This engine rotates at 300 r.p.m. Determine graphically the velocity and acceleration of the crosshead when the crank makes an angle of 45° with the center line of the engine. State numerical values. Scale of drawing $\frac{1}{2}$ size.

16. A steam engine has a stroke of 6 in. and rotates at 300 r.p.m. The connecting-rod is 12 in. long.

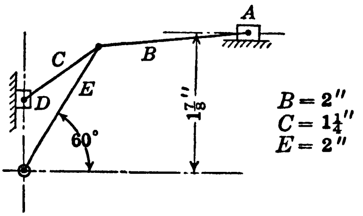
(a) Determine graphically the acceleration of the piston when the crank makes angles of 0°, 60°, and 90° with the center line of the cylinder.

(b) Determine graphically the acceleration of a point on the connecting-rod 3 in. from the crank end, for the same crank positions.

17. A vertical internal-combustion engine has a stroke of $2\frac{3}{4}$ in., and a connecting-rod length of 5 in. This engine runs 2000 r.p.m. clockwise. The center line of the cylinder passes $\frac{3}{8}$ in. to the right of the center of rotation of the crank.

(a) Determine graphically the velocity and acceleration of the piston when the crank makes an angle of 30°, 60°, and 90° with the vertical.

(b) The center of mass of the connecting-rod is 3 in. from the crank-pin. Determine the acceleration of this point when the crank makes an angle of 30°, 60°, and 90° with the vertical.



Problem 18

18. In the figure, determine the acceleration of the slide D if the slide A is moving at a uniform velocity of 10 ft. per sec.

CHAPTER IX

APPLICATIONS OF THE FOUR-BAR LINKAGE

106. Purpose. It is the purpose of this chapter to discuss, more or less in detail, some of the common linkages and their applications to machines and instruments used in engineering practice.

107. Levers, Rocker Arms, Bell Cranks. An oscillating crank is commonly called a lever. Two oscillating levers rigidly connected at an angle of approximately 180° and oscillating about the same center constitute a rocker arm. When the angle between the levers is 90° or less, the combination is called a bell crank. Illustrations of several types are given in Fig. 96.

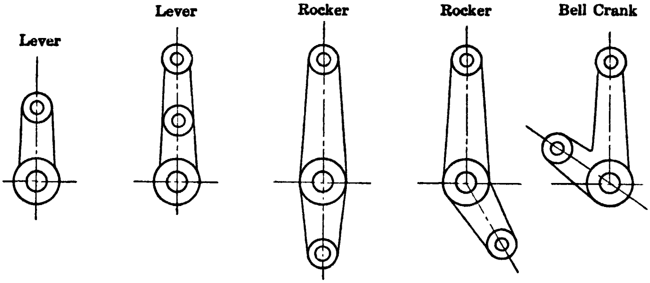


Fig. 96

Any one of these three links may be used to change either the direction or the magnitude of motion of reciprocating parts. The distances traversed and the linear speeds of the ends of the levers are of course proportional to their distances from the center of rotation.

108. Parallel Motion. A simple four-bar linkage having two equal cranks and a connecting-rod equal in length to the distance between the fixed centers forms a parallelogram. The two cranks are always parallel; and the driven crank always has the same angular velocity as the driving crank.

The side-rod connecting all the driving wheels on one side of a locomotive is a common example of this linkage. In this case, power is supplied from the engine cylinders to one of the driving wheels. The motion of this wheel is transmitted to the other driving wheels through the side rods. Figure 97 shows this mechanism.

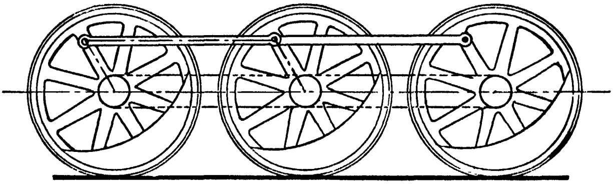


FIG. 97

109 Universal Drafting Machine. Draftsmen often use the mechanism illustrated in Fig. 98 in place of a T square and triangles. The points A and B are fixed to the drafting board.

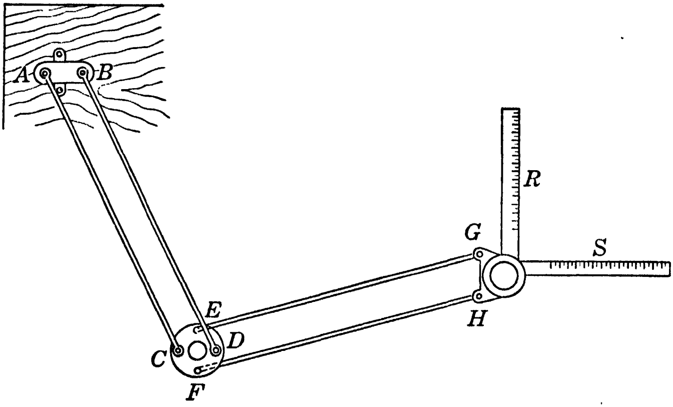


FIG. 98

The points G and H are attached to a movable disc carrying graduated straight edges.

$$AB = CD, \quad AC = BD, \quad EF = GH, \quad EG = FH.$$

The link CD always remains parallel to the fixed link AB. The line EF makes the same angle, usually 90°, with CD. Similarly, GH remains parallel to EF, and the straight edges R and S can move over the board, parallel to their original positions.

110. Pantograph. Figure 99 shows a parallel linkage with an additional link added. Point A is fixed. The links AB, EF, and DC are equal. The distance BF equals AE, and FC equals ED.

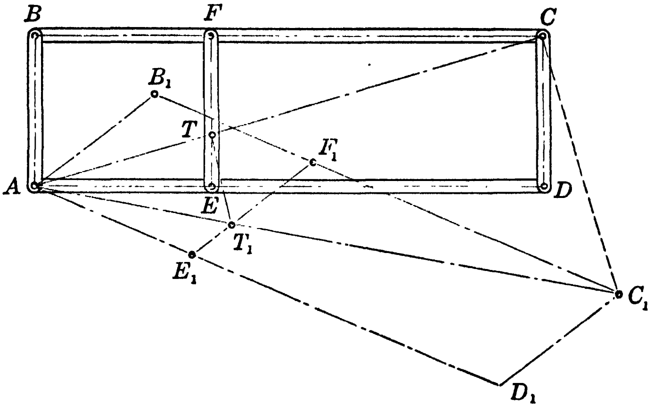


FIG. 99

A line connecting A and C always intersects EF at the same point, T, and the ratio AT/AC remains constant. If C is made to follow a given path, T will duplicate this path to a smaller scale.

PROOF. Assume that C is moved over any path to C_1, the pantograph taking the position $AB_1C_1D_1$. Draw AC_1 intersecting E_1F_1 at T_1. From similar triangles

$$\frac{ET}{AE} = \frac{DC}{AD}, \quad \text{and} \quad \frac{E_1T_1}{AE_1} = \frac{D_1C_1}{AD_1}.$$

But $AE = AE_1$, $DC = D_1C_1$, $AD = AD_1$; therefore $ET = E_1T_1$. Hence a line from A to C always intersects EF in the same point T. The line CC_1 is parallel to TT_1 since

$$\frac{AT_1}{AC_1} = \frac{AE_1}{AD_1} = \frac{AE}{AD} = \frac{AT}{AC} = \frac{TT_1}{CC_1}.$$

The displacement of T always being parallel and proportional to the displacement of C, it follows that T will trace a path duplicating that of C to a smaller scale.

The pantograph is used for reducing or enlarging drawings, pictures, and complicated designs. It is also used as a guide for cutting-tools for copying and enlarging, in wood, stone, or metal, models and designs cut in wood or other easily-worked material. This includes the duplicating of relief designs and carvings on furniture, duplicating of metal dies, etc. Various forms of the pantograph are shown in Fig. 100.

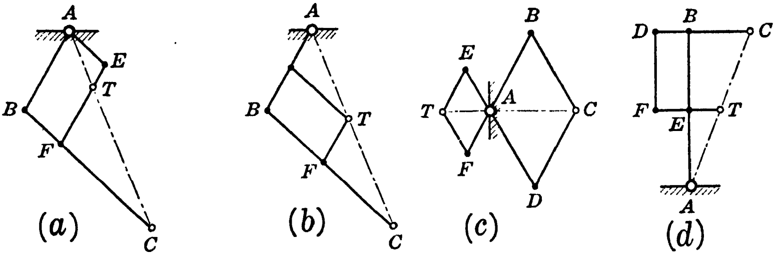


FIG. 100

A form of pantograph called the "lazy tongs" is used for telephone racks, short-lift elevators, elevating platforms, etc.

111. Boehm's Coupling. Two discs or their equivalent bell cranks, fixed to parallel shafts and connected by two or more parallel links, constitute a mechanism used to transmit positive motion, from one shaft to another, the two shafts revolving at the same angular velocity. The planes of the discs must be offset, as shown in Fig. 101, to enable the links to pass each other.

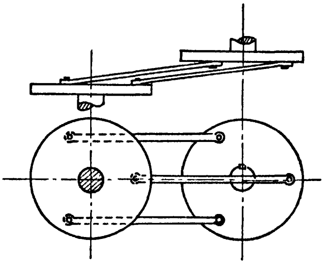


FIG. 101

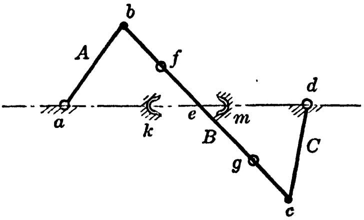


FIG. 102

112. Crossed Links. Two equal cranks and a connecting-rod equal in length to the distance between the fixed centers, but crossing the center line, are used to connect two parallel shafts rotating in opposite directions. Some provision must be made to keep the cranks turning in opposite directions when passing through the dead positions. Figure 102 shows this linkage with the pins *f* and *g* and their mating members *m* and *k* (attached to the supporting frame) provided for this purpose. Properly arranged springs, gear sectors, or extra links may also be used. In § 55, it is shown that

$$\frac{\omega_A}{\omega_C} = \frac{de}{ae}.$$

As *A* rotates, the point *e*, where the connecting link crosses the line of centers *ad*, moves along the line *ad*. Therefore the velocity ratio of the cranks is not constant; and if *A* rotates with uniform angular velocity, *C* will have a constantly varying angular velocity. Both cranks make one complete revolution in the same time.

This linkage is seldom used. It is shown in § 182 that the same relative motion of the shafts can be obtained by means of elliptical gears, and usually these are used.

113. Drag Link. Figure 103 shows a linkage used to transmit rotation between two parallel shafts. Both shafts rotate in the same direction, but the velocity ratio is not constant.

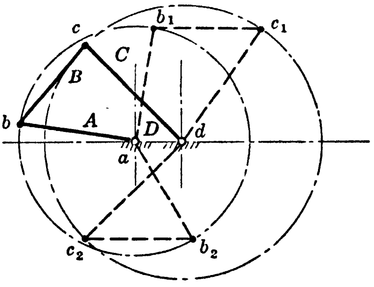


FIG. 103

When the driving crank *A* rotates counter-clockwise from the position ab_1 (where the link *B* is parallel to the fixed link *D*) to the position ab_2 (where these two links are again parallel), the angle b_1ab_2 traversed by the driving crank is larger than the angle c_1dc_2 traversed by the driven crank. During this portion of the cycle, the crank *C* is rotating more slowly than the crank *A*. During the portion of the cycle in which *A* rotates counter-clockwise from ab_2 to ab_1, the crank *C* is moving faster than the crank *A*.

The rapid motion of C while A rotates from ab_2 to ab_1 may be taken advantage of in giving a quick return-stroke to the cutting tool of slotters and shapers, but has not been used to any great extent. Figure 104 shows the drag link arranged for this use.

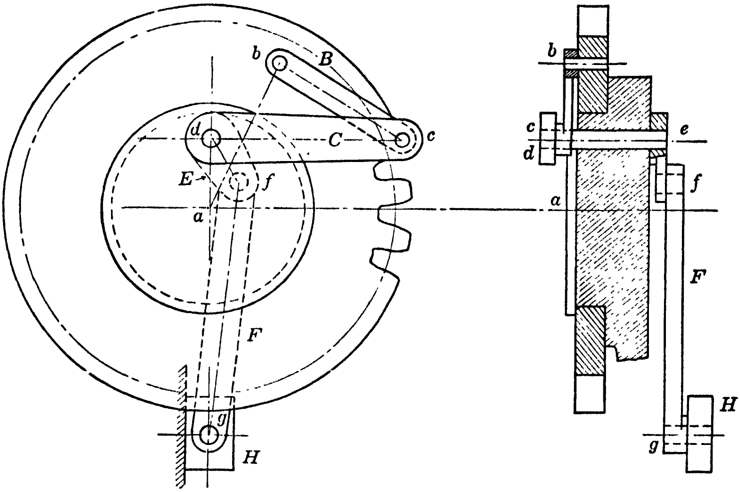


FIG. 104

The large gear is the driving crank A, rotating at uniform angular velocity. The crank C rotates on a bearing inside of the boss on which the gear rotates. On the shaft keyed to C is another crank E, connected by a link F to the slide H which carries the cutting tool.

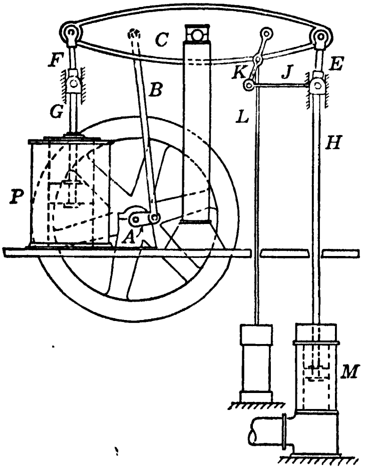


FIG. 105

114. Beam Engines. Figure 105 shows an arrangement used for pumping. A is a rotating crank driven from the crank shaft of the engine. A is connected to the oscillating rocker C by the rod B. The rod E is connected by the rod H to the pump plunger M, giving the plunger a

reciprocating motion. In many pumps of this type, the steam piston and cylinder are placed as shown at *P*. Notice the pantograph *CEJKL*, used to guide the upper end of the rod *L* in a straight path parallel to the path of the upper end of *H*.

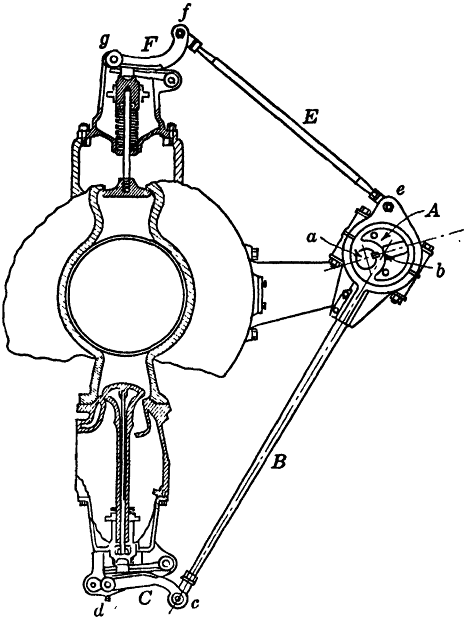


FIG. 106

115. Poppet-Valve Gear. Figure 106 shows the valve driving-gear used on large steam and gas engines. The driving eccentric *A* corresponds to a rotating crank *ab*. The rod *B* causes the lever *C* to oscillate. One side of *C* forms a cam which transmits the motion to the valve proper. The upper valve has a somewhat different motion, since the center line of the rod *E* does not pass through the center of the eccentric at *b*.

116. Quick-Return Linkage. In the case of a rotating crank driving an oscillating link or lever, the time required for the forward stroke of the oscillating lever is different from the time required for the return stroke. By proper arrangement of the parts, any desired time ratio can be obtained. A number of solutions for this problem can be obtained, the particular solution depending on the arrangement of the other machine members and on the space available. The best solution can be obtained by a study of §§ 29 and 30, and by a trial and error method of layout. For example, assume that the driving crank is to rotate at uniform angular velocity, and that the time required for the forward stroke of the lever is twice that required for the return stroke.

In Fig. 107, divide the crank-pin circle, or path of b, into three equal divisions. The crank-pin is to move two divisions clockwise from b_1 to b_2 during the forward stroke of the oscillating lever. When the crank is in position ab_1 or ab_2, the lever will be at the end of its stroke, and the crank and connecting-rod will lie in the same straight line.

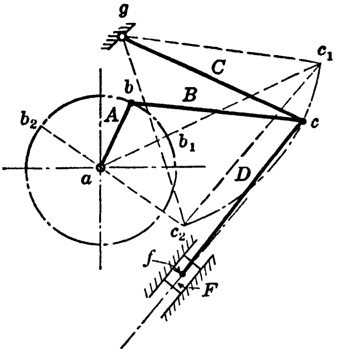


Fig. 107

Assume any desired length of connecting-rod B. Lay off b_1c_1 and b_2c_2 on ab_1 and ab_2 extended, and equal to the desired connecting-rod length. Then gc_1 and gc_2 will be the end positions of the oscillating lever. The center of rotation, g, of this lever will lie on the perpendicular bisector of c_1c_2. The length of the lever is determined by the angle through which C is to oscillate.

This linkage can be used as a quick-return motion for cutting tools by connecting the sliding tool-holder F to the oscillating lever by means of a rod D.

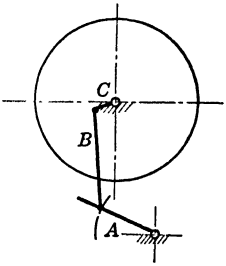


Fig. 108

117. Treadle Motion. In Fig. 108, the oscillating lever A is the driver. This form of the linkage is the common foot treadle used to drive grindstones, sewing machines, etc., where oscillating motion of A is used to produce continuous rotating motion of C.

118. Oscillating-Arm Quick-Return Motion. Figure 109 shows another quick-return linkage used in shapers and similar machine-tools to give a slow speed to the cutting tool during the cutting stroke and a high speed during the return stroke. The crank A is driven at a uniform velocity by gearing or belts applied to the shaft at a. The slotted link, C, oscillates on the fixed center c, causing the slide E to reciprocate on its guides. The cutting-tools are attached to the slide E.

The slide *E* is at the end of its stroke when the crank *A* and slotted link, *C*, are perpendicular. During the cutting stroke of *E*, the crank turns through the angle θ, and during the return stroke the crank turns through the angle ϕ. Since the crank rotates at uniform velocity, the time-ratio for the two strokes is

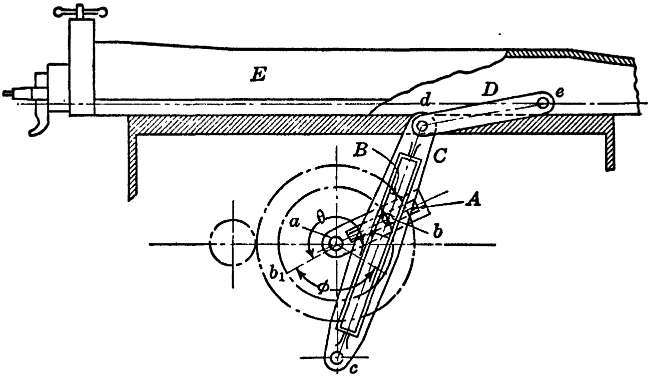


FIG. 109

$$\frac{\text{time of cutting stroke}}{\text{time of return stroke}} = \frac{\theta}{\phi}. \tag{61}$$

If the distance *ac* is adjusted, any desired ratio can be obtained; in Fig. 109, the ratio is 2 to 1. Suppose a stroke-ratio of 3 to 1 is desired. Lay off the angle cab_1, equal to 180° divided by the ratio plus 1, or

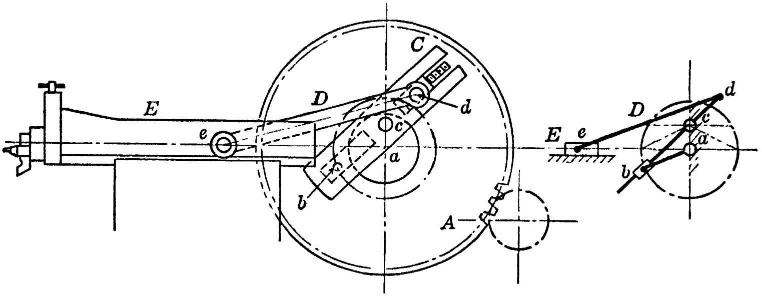
$$\frac{180^\circ}{3+1} = 45^\circ = \frac{\phi}{2}.$$

Draw b_1c perpendicular to ab_1, cutting the line of fixed centers at *c*, locating the fixed center about which the link *C* oscillates. The crank *A* will travel 6 times 45°, or 270°, equals θ, during the cutting stroke, and 2 times 45°, or 90°, equals ϕ, during the return stroke. The cutting stroke ratio is

$$\frac{\text{cutting stroke}}{\text{return stroke}} = \frac{270^\circ}{90^\circ} = 3.$$

If the length of the link *C* is varied, the stroke of the slide *E* may be varied also. The stroke is usually varied, however, by altering the length of the crank *A* by means of a screw attached to the slide block *B*.

119. Whitworth Quick-Return Motion. This linkage shown in Fig. 110a is of the same type as that of Fig. 109, except that the crank *ab* is made longer than the center distance *ac*. This causes the link *C* to rotate about *c* instead of oscillating.



(*a*) (*b*)

FIG. 110

If the path of the slide, *E*, passes through the fixed center *c* perpendicular to the line of centers *ac*, Fig. 111, the stroke time ratio is

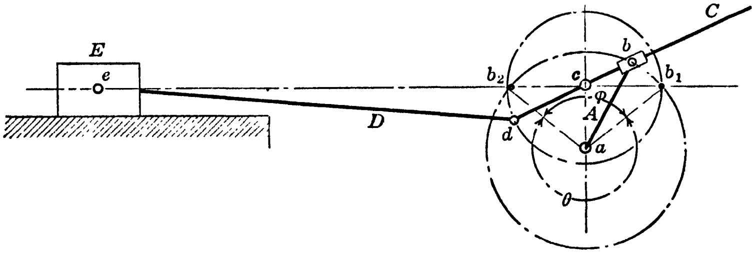


FIG. 111

$$\frac{\text{time of cutting stroke}}{\text{time of return stroke}} = \frac{\theta}{\phi}.$$

The stroke of the slide may be varied by adjusting the length of the arm *C*, that is, the distance *cd*.

120. Oscillating Engine. In Fig. 112, the cylinder of the engine is supported on trunnions at *d* instead of being rigidly fixed to the frame. As the piston moves in the cylinder, the crank rotates on the center, *a*, and the cylinder oscillates on the trunnions at *d*. This design is used on small engines with the valve ports in the trunnion of the cylinder. Steam enters from a stationary chamber fitting over the trunnions, the ports being opened and closed by the oscillation of the cylinder. No valve linkage is necessary.

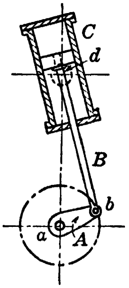


Fig. 112

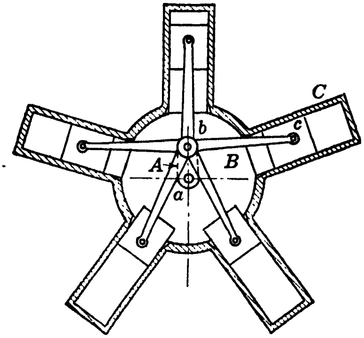


Fig. 113

121. Rotary Engine. If the crank *ab*, of the ordinary engine is held stationary, as in Fig. 113, the cylinders will rotate about the center *a*, corresponding to the crankshaft of the engine. Several multiple-cylinder rotary engines such as the Gnome, used in airplane service, operate on this principle.

122. Toggle-Press. A crank and crosshead linkage with the driving crank rotating with uniform velocity gives the crosshead a very slow movement when near the dead positions. This slow movement can be used to exert high pressures on materials placed between the crosshead and the fixed head, as in Fig. 114. This is sometimes loosely called a toggle-press and is used on light presses or on presses where only the last part of the stroke is required for pressing or forming work.

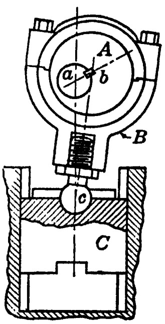


Fig. 114

The true toggle mechanism consists of two links connected by a knee or forked joint that may be brought in line or nearly in line by some outside force and made to produce a great endwise pressure. A common arrangement for such a linkage is to pivot one link, having the other link end in a slide, and to operate them by means of a crank and connecting-rod, as shown in Fig. 115. This form of toggle is used on presses, punches, riveters, and crushers requiring a great force during most of the stroke.

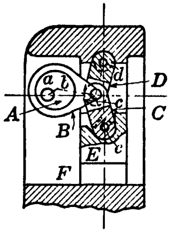


FIG. 115

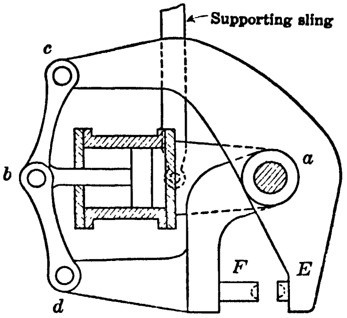


FIG. 116

A good example of the toggle mechanism as used on lever-riveters for bridge-building work is shown in Fig. 116. Moderate air-pressures or water-pressures are used in the cylinder to produce high pressures on the rivets placed between the rivet dies at *E* and *F*.

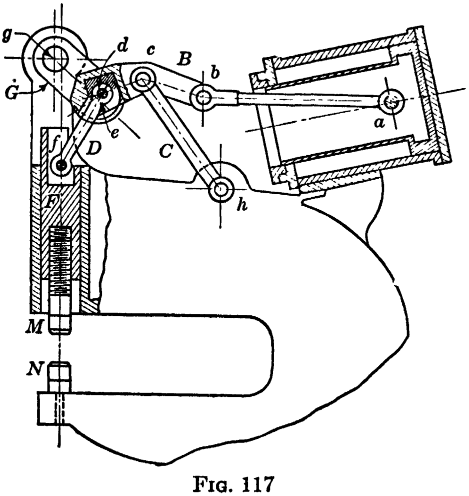


FIG. 117

The Hanna riveter shown in Fig. 117 uses a more complicated linkage called a double toggle. Air pressure applied to the piston causes a fairly rapid motion of the slide *F* during the first part of the stroke, with gradually increasing pressure between the slide and the fixed head, *N*, and slower movement during the last part of the stroke with nearly uniform high pressure between *M* and *N*.

123. Corliss Valve-Gear. The principal object in the design of this linkage is to obtain a very slow movement of the valve lever *K*, Fig. 118, during a large part of the cycle, and a rapid movement during the remainder of the cycle, this part of the cycle being that during which the valve is uncovering the steam-ports to the engine cylinder. Motion is derived from an eccentric *ab* on the engine shaft, equivalent to a crank *A* rotating at uniform velocity. The eccentric through the eccentric-rod, *B*, rocking lever, *C*, and reach-rod, *E*, causes the pin *f* on the wrist-plate *F* to oscillate from f_1 to f_2 about the center *g*, with approximately harmonic motion. The valve-rod *H* connects the valve-lever *K* to the wrist-plate.

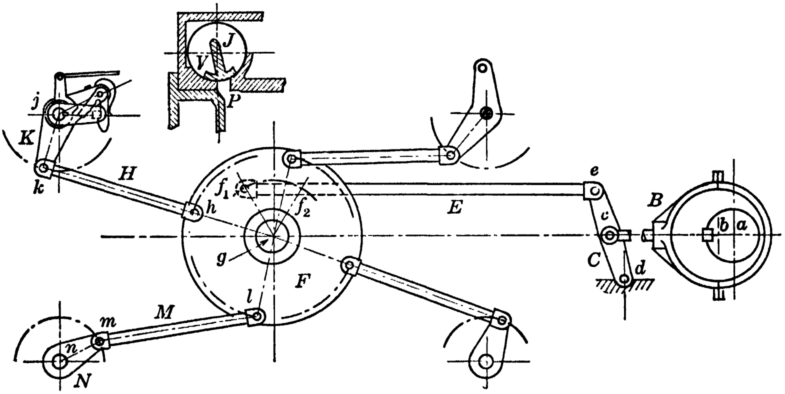


Fig. 118

The linkage is proportioned so that when *f* is in its extreme position, f_1, the points *g*, *h*, and *k* are very nearly in a straight line perpendicular to *jk*. As *f* moves from f_1 towards f_2, very little motion is transmitted to *K* during the first part of the movement, but during the last part of the movement *K* moves rapidly. It is during this part of the cycle that the valve *V*, attached to the lever *K*, moves across the port *P*, admitting steam to the cylinder. The valve is connected to the lever *K*, by a trip-mechanism, allowing the valve to be released by the governor, and quickly returned to its starting position by a dash-pot (not shown). When the lever *K* completes its cycle, it is again connected to the valve, and the cycle repeated. The exhaust-valve is connected to the wrist-plate by the rod *M* and lever *N*. No trip-mechanism is used on the exhaust-valve.

124. Armstrong Non-Releasing Valve-Gear. On account of the releasing mechanism, the use of the Corliss gear is limited to slow-speed engines. On high-speed engines, a linkage similar to Fig. 119 is used to operate the valve. The valve-lever *A* is keyed to the valve-stem. The lever *C* rotates on the valve-stem, but is not attached to it. The bell-crank *EK* is driven by an eccentric on the engine shaft and oscillates with approximately harmonic motion.

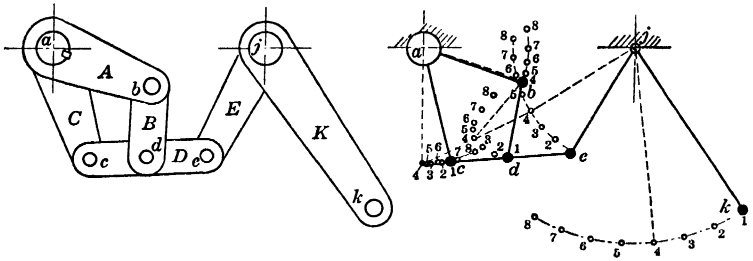


FIG. 119

During the first part of the movement of *K*, the link *B* rotates on the center *b*, giving very little motion, if any, to the valve-lever *A*. During the last part of the movement the valve-lever *A* moves rapidly, causing the valve to open quickly. There is no trip or releasing mechanism used with this valve-gear, and the valve returns to its starting position with the return oscillation of the driving crank *K*.

PROBLEMS

1. Two slides travel on parallel guides 6 in. apart. They are each connected to an oscillating lever.

(a) Locate the center of the lever so that both slides will move in the same direction, one moving 2 in. while the other moves 1½ in.

(b) Locate the center of a rocker-arm so that the slides will move in opposite directions.

2. Arrange a bell-crank connected to two slides by 3-in. links so that one slide moves 2 in. in a horizontal direction and the other 3 in. on a 45° line.

3. A slide *A* reciprocates on a horizontal line over a path 2 in. long. It is connected by two rods and a rocker-arm to a second slide moving in the opposite direction over a path 3 in. long. If the paths are parallel and 2½ in. apart, lay out the connecting linkage.

4. Lay out a universal drafting-machine which, when fastened at the center of the top edge of a 40-by-54-in. board, can reach all points on the board with the center of the head holding the scales.

5. A lever-arm oscillates through an angle of 30° on each side of the center position. Determine the length of arm to give the least vibration to a link connecting this arm to a sliding block moving on a line perpendicular to the central position of the lever-arm, and 2 in. from the center of oscillation.

6. A beam-engine has a stroke of 4 ft., and the cylinder center-line is 6 ft. from the center-bearing of the beam. What is the best length of the beam?

7. Find all the dimensions of a pantograph reducing motion for use on a steam engine having a stroke of 16 in. The stroke is to be reproduced to ¼ size. The distance from the cylinder center-line to the fixed point of the pantograph is 20 in.

8. Design a pantograph to be connected to a crosshead of a steam engine to reproduce the crosshead motion to a small scale. The crosshead has a 12-in. stroke, and the movement is to be reduced to 3 in. The fixed point of the pantograph is to be 16 in. above the crosshead center line. Give all dimensions.

9. Lay out a pantograph which will be capable of enlarging a 2-in. circle to a 5-in. circle. The minimum distance from the tracing point to the fixed center is to be 1 in. Give all dimensions.

10. A 6 in. x 10 in. drawing is to be enlarged to 18 in. x 30 in. Design a suitable pantograph, and give all dimensions.

11. Design a pantograph that may be used to reproduce sketches so that the size of the reproduction will be 2½ times as large as the original. The original sketches are of such size that they will fall within a 4-in. circle. Indicate the fixed center, and give all dimensions.

12. A pantograph is to be used to enlarge drawings included in a 4-in. square to three times their original size. Lay out this pantograph and give all dimensions.

13. A revolving crank 2 in. long is to drive a lever oscillating through an angle of 60°. The distance between fixed centers is 3 in. Lay out a four-bar linkage for this purpose.

14. Two cranks, each 1½ in. long, are connected by a link 3 in. long, and the distance between fixed centers is 3 in. The shafts are to rotate in opposite directions. Plot a curve showing the variation in the ratio of the angular velocity of the driver to the angular velocity of the driven. Plot values of the ratio as ordinates to the scale 1 in. equals 1, and the angular positions of the driver as abscissae to the scale 1 in. equals 60°. Start with the driver on the center-line between the fixed centers.

15. Two cranks *A* and *B*, each 2 in. long, rotate about fixed centers 5 in. apart. They are connected by a 5-in. link crossing the line of centers. The crank *A* rotates with uniform angular velocity.

(a) Determine the ratio of the angular velocity of *B* to the angular velocity of *A* when the crank *A* is perpendicular to the center-line.
(b) Determine the maximum and minimum angular-velocity ratios.

16. A drag-link consists of a driving crank 1½ in. long and a driven crank 2 in. long, connected by a link 2 in. long. The distance between fixed centers is ½ in. Plot a curve showing the variation in the angular velocity ratio of the cranks for a complete revolution of the driver. Make the scale for the ordinates 1 in. equals unit velocity ratio, and for the abscissae, 1 in. equals 120° movement of the driver. Start with the cranks in the position of unit-velocity ratio. Indicate on the drawing the position of the driven crank when it is moving fastest and slowest.

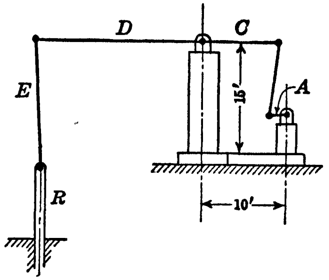
17. A drag-link consists of a uniformly rotating driving crank *A*, 1½ in. long, a driven crank *B*, 2 in. long, and a connecting-link, 2 in. long. The fixed centers are 1 in. apart.

(a) Draw this linkage in the position in which the crank *B* has its maximum angular velocity.
(b) Draw this linkage in the position in which *B* has its minimum angular velocity.

18. The drag-link of Problem 17 is to drive the tool-holder of a key-slotting machine whose stroke is 1½ in., by means of a crank and connecting-rod. The connecting-rod is to be 6 in. long. Determine the ratio of the cutting stroke to the return stroke of the tool-holder.

19. A crank 3 in. long is to drive an oscillating-lever by means of a connecting-rod 8 in. long so that the forward oscillation will take twice as long as the return oscillation. The lever is to oscillate through 60°. Locate the fixed center of the lever, and determine its length.

20. In the figure shown, the crank of an oil-pumping rig is 2 ft. long, and the rod E is 8 ft. long. Arrange this linkage so that the stroke of the pumping-rod will be 8 ft. Give complete dimensions. Through what angle will the rocker-arm DC oscillate?



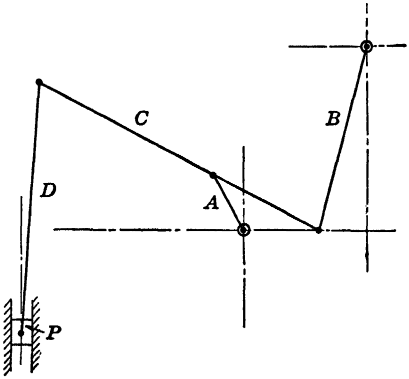
PROBLEM 20

21. Design a quick-return motion similar to Fig. 107 so that the driven lever will oscillate through an angle of 60° and make the return stroke in ⅔ of the time required for the forward stroke. The driving crank is to be 1½ in. long.

22. Lay out a system of links and levers which will have a driving crank turning through a complete revolution while a driven lever makes two complete oscillations.

23. In the gas-engine valve drive shown in Fig. 106, plot a curve showing the angular movement of the arm F for one complete revolution of the eccentric. Plot angular movement of F as ordinates to the scale 1 in. equals 15°, and the angular movement of the eccentric as abscissae to the scale 1 in. equals 120°. Start with ab in the vertical position. Point g is 27 in. to the left and 20 in. above the point a. Point d is 27 in. to the left and 36 in. below the point a. Point e is 2 in. from the line cb extended.

$$ab = 2\text{ in.},\quad ae = 6\text{ in.},\quad gf = 7\tfrac{1}{2}\text{ in.},\quad ef = 29\text{ in.},$$
$$dc = 9\text{ in.},\quad bc = 43\text{ in.}$$



PROBLEM 24

24. In the pump shown, the up-stroke takes twice as long as the down-stroke. The crank A is 1 in. long and the link B is 3 in. long. Arrange this linkage to give the plunger a stroke of 6 in.

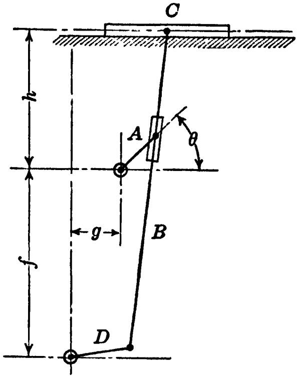
25. Arrange a four-bar quick-return linkage with a rotating crank and oscillating lever to make a return-stroke in one-half the time required for the forward stroke. The crank length is to be 1½ in. and the connecting-link is 4 in. long. Arrange the oscillating lever so that the angle of oscillation is 45°. What is the length of the stroke obtained?

26. In an oscillating-arm quick-return motion for a shaper, the driving crank is 4 in. long, and the time-ratio of the forward-to-return stroke is 3 to 1. Locate the fixed center of the oscillating-arm. If the stroke of the shaper ram is to be 12 in., determine the length of the oscillating-arm.

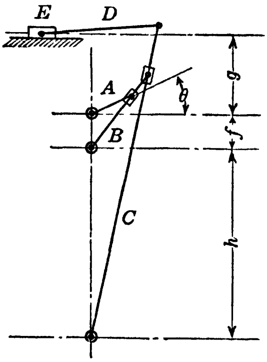
27. The quick-return motion used on an Averbeck shaper is shown in the figure.

$A = 3$ in., $B = 20$ in., $D = 3\frac{1}{2}$ in., $f = 12$ in.,
$g = 3$ in., $h = 8\frac{1}{2}$ in.

(a) Determine the ratio of the cutting time to the return-stroke time.
(b) Determine the stroke of the cutting-ram C.
(c) Determine the velocity of C when $\theta = 45°$, and $60°$, if A runs at 20 r.p.m.



PROBLEM 27



PROBLEM 28

28. The quick-return motion used in the Stockbridge shaper is shown in the figure.

$A = 3$ in., $B = 6$ in., $C = 20$ in., $D = 10$ in,
$f = 2$ in., $h = 12$ in., $g = 4\frac{3}{4}$ in.

(a) Determine the ratio of cutting time to the return-stroke time.
(b) Determine the stroke of the cutting-ram E.
(c) Determine the velocity of E when $\theta = 30°$ and $60°$.

29. Solve Problem 27, substituting a vertical slot in the ram, and a pin in the oscillating-link, for the link D.

30 An Averbeck shaper-driving mechanism is similar to the figure in Problem 27. The fixed center of D is 9 in. below and $5\frac{1}{2}$ in. to the left of the fixed center of A. The pin on the tool ram, C, moves on a horizontal line 19 in. above the center of D. The length of lever D = 6 in., link B = 20 in., crank A = 4 in.

(a) Determine the length of the stroke of the tool ram.
(b) Determine the time ratio of forward to return strokes.
(c) If the maximum cutting speed of e is to be 30 ft. per min., at what speed (r.p.m.) will the driving crank be rotating?

31. Lay out a Whitworth quick-return motion which will give a ratio of time of return stroke to forward stroke of 1 to $2\frac{1}{2}$, using a driving crank 3 in. long, and making the center-line of the tool ram pass through the fixed center of the driven crank.

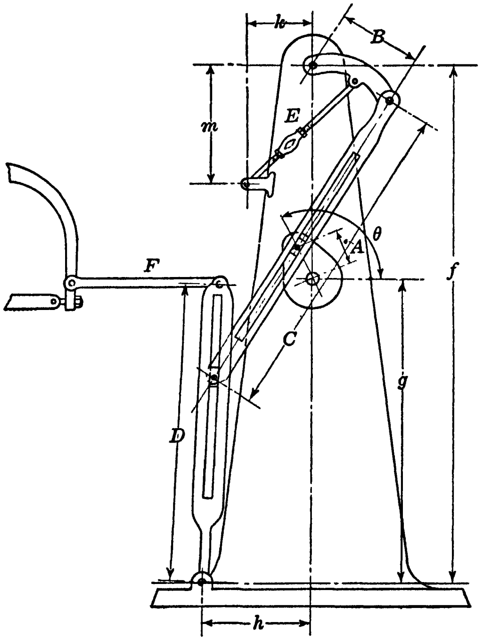
32. A Whitworth quick-return motion is arranged as shown in Fig. 111. The distance $ac = 1\frac{1}{4}$ in., the crank $A = 2$ in., the crank $cd = 1$ in., the link $D = 4$ in. Crank A is the driver and rotates at 30 r.p.m.

(a) Determine the length of stroke of the cutting-ram E.
(b) Determine the time-ratio of cutting stroke to return-stroke.
(c) Determine the maximum speed of E during the cutting stroke.
(d) What is the maximum speed of E during one complete cycle? Indicate the position of the mechanism at the time this maximum speed occurs.

33. The figure shows the drive for a hack-saw

$A = 2$ in., $B = 4\frac{3}{4}$ in.,
$C = 18$ in., $D = 16\frac{1}{2}$ in.,
$f = 28\frac{1}{2}$ in., $g = 16\frac{3}{4}$ in.,
$h = 6$ in., $k = 3\frac{1}{2}$ in.,
$m = 6\frac{1}{2}$ in., $E = 6$ in.

(a) Determine the stroke of the hack-saw, when the link B is 30° below the horizontal.



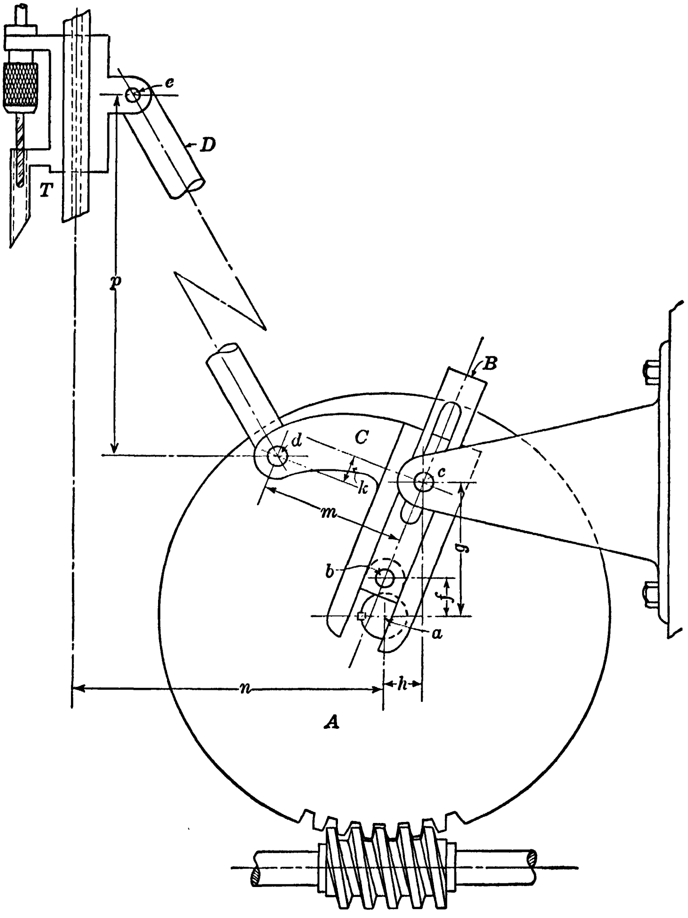
PROBLEM 33

(b) Determine the time-ratio for the cutting stroke and return-stroke.

(c) Determine the velocity of the saw blade when $\theta = 60°$, $90°$, and $120°$, assuming the velocity of the crank pin to be represented by a 1-in. vector.

34. In the mortising machine-drive shown, b is a pin on the gear A, which moves the slide B, in the rocker-arm C, which oscillates on the pin c.

$f = 1\frac{5}{8}$ in. $g = 5\frac{3}{4}$ in. $h = 1\frac{5}{8}$ in. $k = 1\frac{1}{4}$ in. $m = 6\frac{1}{4}$ in. $n = 10$ in. $p = 29$ in.



PROBLEM 34

(a) Determine the stroke of the tool T.
(b) Determine the ratio of the time of the cutting stroke to the return stroke.
(c) Determine the velocity of T when the crank is in the position shown. Assume the velocity vector of b to be 1 in.

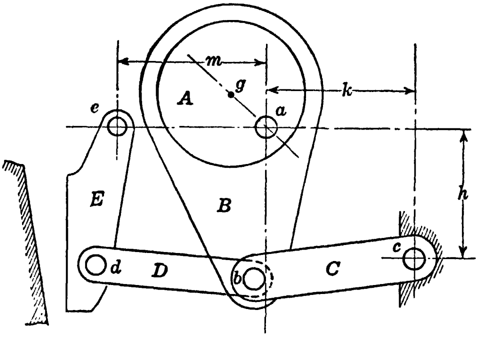
35. In the oscillating engine shown in Fig. 112, assume $ab = 2$ in. and $ad = 8$ in.

(a) Determine the engine stroke.
(b) Determine the total angular movement of the cylinder.
(c) Determine the angular velocity of the cylinder when the crank makes an angle of 30° with the vertical.

36. In the rotary engine shown in Fig. 113, assume the piston stroke to be 3 in. and the connecting-rod to be 5 in. long. Assume that the piston speed is 1200 ft. per min. when in the middle of the stroke, and determine the angular velocity of the engine in revolutions per min.

37. The figure shows the toggle mechanism of a rock-crusher. Assume the velocity of the center of the eccentric (point g) to be represented by a vector 1 in. long.

$ag = 2$ in., $bg = 8$ in., $h = 5\frac{1}{2}$ in., $k = m = 6$ in.,
$bc = bd = 6\frac{1}{4}$ in., $ed = 6$ in.



Problem 37

(a) Plot the velocity diagram of the point d, for a complete revolution of the eccentric A.
(b) When friction is disregarded, the pressure exerted is inversely proportional to the velocity. Plot the force that can be exerted at e, assuming the force applied at g, perpendicular to ag, to be 100 lbs.

38. Consider the mechanism of Fig. 115 to be arranged as a forming press. Disregarding friction, the force exerted by the slide E and the force exerted at b to turn the eccentric are inversely proportional to the respective linear speeds. The press is to be capable of exerting 100,000 lbs. pressure when the eccentric radius ab makes an angle of 15° with the horizontal center-line, and the lines cd and ce make angles of 15° with the vertical center-line. The eccentric radius ab is 1 in., and the distance bc is 10 in. What pressure must be exerted at b perpendicular to ab?

39. An Armstrong valve gear similar to Fig. 119 is used on a four-valve steam engine. The driving lever oscillates 30° on each side of the vertical. The link dimensions are: $A = 3\frac{3}{4}$ in., $B = 3$ in., $C = 4$ in., $D = 4$ in., $E = 4\frac{1}{8}$ in., $K = 6$ in., $cd = 2$ in.

(a) Determine the angular movement of A for each 10° movement of K.

(b) Assuming that the point k has the same linear velocity at the end of the first 10° movement that it has after 50° movement, determine the ratio of the angular velocities of A for these positions of K.

CHAPTER X

CAMS

125. Cams. A cam mechanism is made up of a ***cam,*** a ***follower,*** and the ***frame.*** A cam is a body so shaped or grooved that, by its rotation, oscillation, or reciprocation, it will impart motion to a second body called the follower.

A cam is designed to cause a follower to move over a given path with a given motion in a given time, and it is used extensively to obtain a variety of motions, in special and automatic machinery, that would be difficult to duplicate by other mechanisms. A general classification of cams may be: ***revolving cams, oscillating cams,*** and ***reciprocating cams.*** There are a variety of cams under each of these heads, and it will be necessary further to classify the different types when the general classes are discussed.

126. Follower Motions. Cams are commonly designed to give the follower uniform motion, simple harmonic motion, uniformly accelerated motion, or uniformly retarded motion; any other desired motion, however, may be used.

Uniform velocity is often employed for continuous motion, since it moves the follower over equal spaces in equal intervals of time. For example, suppose a follower A, Fig. 120, is to move from M to N along the vertical center-line with uniform velocity.

From M draw the line $M6'$, making any angle with MN. Lay off six equal divisions on $M6'$. Draw $6'N$, and draw lines through $5'$, $4'$, $3'$, etc., parallel to $6'N$, cutting MN at 5, 4, 3, etc. Then MN will be divided into six equal spaces, the points 1, 2, etc. indicating the position of the follower point at the end of equal time intervals.

Simple harmonic motion can be shown graphically by the application of the definition.* For example, suppose a follower A,

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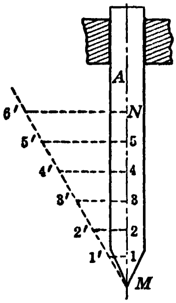
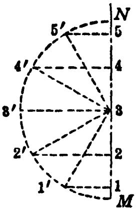
* If a point moves along the circumference of a circle with uniform velocity, the projection of that point on a diameter of the circle moves with simple harmonic motion.

Fig. 121, is to move from *M* to *N* along the vertical center-line with simple harmonic motion.



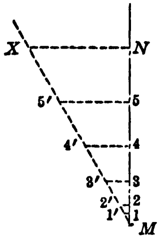
Uniform motion

Fig. 120



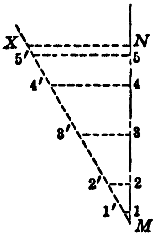
Simple harmonic motion

Fig. 121



Uniformly accelerated motion

Fig. 122



Uniformly accelerated and retarded motion

Fig. 123

On *MN* as a diameter, draw the semi-circle *M3'N*. Divide the semi-circle into any number of equal parts, in this case six, and project these parts, *1'*, *2'*, *3'*, etc., onto the diameter *MN*, finding the points *1*, *2*, *3*, etc. According to the definition of simple harmonic motion, if the body *A* moves from *M* to *1* in one unit of time, from *1* to *2* in the second unit of time, etc., it will traverse the space *MN* with simple harmonic motion.

Uniformly accelerated motion over any given space can be laid out as follows:

If a uniformly accelerated body starts from rest, the distance passed over for any given time can be calculated from the formula $s = at^2/2$, where s = space passed over, a = acceleration (which is constant), and t = time.

The acceleration being constant, the space traversed will be proportional to the square of the time. The first unit of time corresponds to one space unit, the first two time units correspond to four space units, the first three time units correspond to nine space units, and so on indefinitely. The total spaces traversed will be 1, 4, 9, 16, 25, . . . etc., units, and the successive spaces traversed in equal time units will be in the proportion 1, 3, 5, 7, 9, . . . etc., each interval being two space units larger than the preceding space interval.

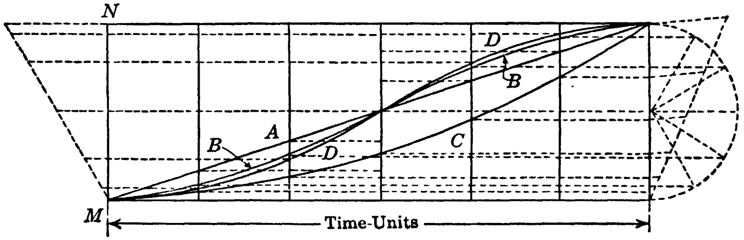
Figure 122 gives the method of laying out the space *MN* if a follower *A* moves from *M* to *N* with uniformly accelerated motion.

Uniform retardation, or decelerated motion, is simply the reverse of uniformly accelerated motion, so that the spaces traversed in equal time intervals will be proportional to . . . 9, 7, 5, 3, 1.

For example, a follower, *A*, is to move from *M* to *N*, Fig. 123, with uniformly accelerated and retarded motion, and the divisions of the space *M–N* to give this motion may be found as follows.

Draw *MX* making any angle with *MN*. Assume the number of divisions (in this case 6), and a unit of length. Make *M1′* equal to 1 unit, *1′–2′* equal to 3 units, *2′–3′* equal to 5 units, and so on, dividing the distance *MX* proportional to 1, 3, 5, 5, 3, 1. Draw *XN*, and from *5′*, *4′*, etc., draw lines parallel to *XN*, cutting *MN* at *5*, *4*, *3*, etc.

The points *1*, *2*, *3*, . . . etc., will locate on the line *MN* the position of the follower at the end of equal time intervals as it moves with uniformly accelerated and retarded motion.



A—Uniform velocity
B—Simple harmonic motion
C—Uniformly accelerated motion
D—Uniformly accelerated and retarded motion

Fig. 124

From the space-time diagrams drawn in Fig. 124 for each of the motions just discussed, it will be noticed that uniform motion gives an abrupt change from zero velocity to maximum velocity and vice versa. The diagram, therefore, is usually made with easement curves. The ***easement curves*** provide a short period of approximately uniform acceleration at the beginning and a like period of approximately uniform deceleration at the end of the follower motion, thus bringing the follower up to velocity, and from velocity to rest, without the abrupt changes due to uniform velocity. Uniformly accelerated and retarded motion allows a body to start and stop with ease, and for that reason it is often employed. Simple harmonic motion falls in between the other two motions and gives results very similar to uniformly accelerated and retarded motion.

The motions here discussed are by no means the only ones that it is possible to use in cam design. Those discussed follow definite laws and are easily constructed. Any desired motion can be laid out by assuming the space-time diagram and projecting back onto the path to be divided.

127. Revolving Cams. Any cam that revolves in order to impart motion to a follower is known as a revolving cam. Most cams are of this class. In order to simplify the discussion of revolving cams, they may be classified according to their general form, that is, the ***plate cam*** made from a flat plate, with the edge operating the follower; the ***cylindrical cam*** made from a cylinder, with a groove or strap operating the follower; the ***conical cam*** made from a cone, with a groove operating the follower; and the ***spherical cam*** made from a sphere, with a groove operating the follower.

Plate cams and cylindrical cams are in common use, and a detailed discussion of these cams will be taken up in succeeding articles. Conical cams and spherical cams are seldom used, and only a mention of their possible field of application is included in this text.

The student should bear in mind that the classification of cams here included is not rigid, and the various forms that a cam may take are limited only by the imagination of the designer and the motion required of the follower.

128. Plate Cams. Plate cams are designed to operate followers, which may be ***pointed, flat faced,*** or equipped with a ***roller.*** Also, any of the above followers may be made to reciprocate, oscillate, or move along an irregular path. Special plate cams (so called because of their design or layout characteristics) may be classified as follows:

A. Positive motion plate cams.
    (1) Grooved plate cams.
    (2) Constant diameter cams.
    (3) Main and return cams.
    (4) Constant breadth cams.
B. Involute cams.

The meaning of each of the above terms will be made clear by the following articles and examples.

129. Plate-Cam Design. In the design of a plate cam, it is necessary to lay out the curve of the cam so that it will cause one of the types of followers mentioned above to move over a given path with a definite motion, or to lay out the curve of the cam according to some definite curve such as the involute. The motion imparted to the follower may be one of those previously described, or any other desired motion.

The design can best be studied if the different types of cams are laid out. A number of examples follow to illustrate some of the principles involved as well as methods of construction.

In the examples, the path of the follower has been divided into a small number of parts for the sake of clearness. The actual cam layout would require many more divisions for accuracy, especially if the required motion of the follower is irregular or is rapidly changing in its nature.

130. Point Follower. Consider a shaft *A*, Fig. 125a, and a point follower *B*. The follower is located so that its center-line passes through the center of the shaft, and its point is one unit above the shaft center when in its lowest position.

Required. Lay out a plate cam that may be keyed to the shaft *A* and that will cause the follower *B* to rise 1½ units with uniform motion while the cam rotates in a clockwise direction through 180°, and to return to its original position with simple harmonic motion during the next 180° of rotation.

Construction. First draw the base circle, Fig. 125b, which has a radius equal to the distance from the center of the shaft to the point of the follower, when the follower is in its lowest position. In this example, the base circle has a radius of one unit, and the follower will be in contact with the base circle at the point *O*. Next lay off a distance of 1½ units up from *O* to *4* on the center-line of the follower (the distance the follower is to be raised), and divide this distance *O–4* into any number of equal parts, say four divisions. The follower is to be raised while the cam is rotating through 180°. Therefore start with *AO* and lay off 180° opposite to the direction of rotation, in this case to the left, and divide this arc *O–X* into four equal divisions.

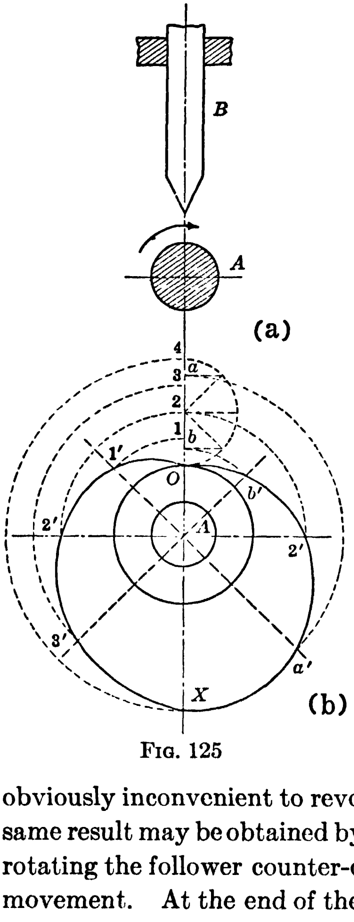


Fig. 125

The path of the follower is now divided into spaces traversed in equal time intervals. While the point of the follower rises to point *1*, the cam will turn clockwise so that the radial line *A1′* will be in the position *AO1*, and the point *O* on the cam surface will rotate clockwise through the same angle. It is obviously inconvenient to revolve the cam in the drawing, but the same result may be obtained by considering the cam stationary and rotating the follower counter-clockwise through the same angular movement. At the end of the first time interval, the point of the follower will be at point *1′* instead of at point *1*; therefore point *1′* must be a point on the cam outline. Points *1′*, *2′*, etc. may be located by the following construction.

With *A* as a center and *A1* as a radius, strike an arc cutting the first division line at *1′*. In like manner *2′*, *3′*, and *4′* are found. To lower the follower from *4* to *O*, divide this distance for harmonic motion (as previously explained), using four divisions.

Divide the remaining 180° of the base circle into four equal divisions (always the same number as was used on the follower path). With *A* as center and radius *Aa,* strike an arc cutting the first division line at *a′*, and find the points *2′* and *b′* in the same manner. Draw a smooth curve through these points forming the required cam outline.

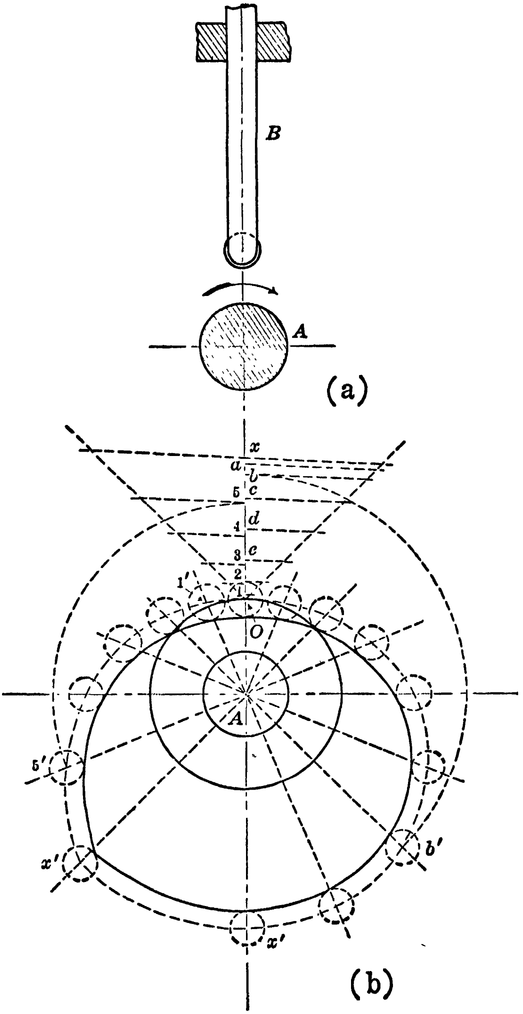


FIG. 126

131. Roll Follower. The pointed or sharp-edged follower wears rapidly and also tends to cut into the cam face. To relieve this condition, a roll follower is frequently substituted for the point follower, thus reducing both wear and frictional losses.

GIVEN. A shaft *A*, Fig. 126a, and a roll follower *B,* such that the center-line of the follower passes through the center of the shaft, and the center of the roll is one unit above the shaft center when in its lowest position.

REQUIRED. Lay out a plate cam that may be keyed to the shaft *A* and that will cause the follower *B* to rise $1\frac{1}{2}$ units with uniformly accelerated motion while the cam rotates in a clockwise direction through 135°, to remain at rest during the next 45° of the cam's rotation, and to return to its original position with uniformly accelerated and retarded motion during the remaining 180° of the cam's rotation.

Construction. To design this cam, first consider the follower as a point follower, using the center of the roll as the point. Draw the base circle, Fig. 126b, with *A* as center and a radius of one unit, which makes the base circle pass through the center of the roll. Lay off *Ox* equal to 1½ units, the distance the follower is to be raised, and divide *Ox* into six divisions such that its motion will be uniformly accelerated, locating the points *1*, *2*, *3*, etc. and *x*. Lay off the angle *OAx'* opposite to the direction of rotation equal to 135°, and divide this angle into six equal parts. With *A* as center and radius *A5*, draw an arc cutting the corresponding division line at *5'*, and in like manner locate the points *1'*, *2'*, *3'*, *4'*, and *x'*. Since the follower is to remain at rest for the next 45° of rotation, the portion of the curve subtended by this angle will be an arc of a circle whose radius is *Ax*. The follower is to be lowered with uniformly accelerated and retarded motion; so divide *xO* into eight parts for this motion locating points *x*, *a*, *b*, *c*, *d*, *e*, etc. Divide the remaining 180° of the base circle into eight equal parts. With *A* as center and radius *Ab*, draw an arc cutting the corresponding division line at *b'*. In the same manner locate the points *a'*, *c'*, *d'*, *e'*, etc. A curve drawn through these points will be the outline of a cam for a point follower, and is called the ***theoretical cam curve*** or ***pitch curve***. Use this curve as a locus of centers, and draw the roll any number of times. The curve drawn tangent to these roll arcs will be the required cam outline.

132. Off-Set Roll Followers. Consider a shaft *A*, Fig. 127a, and a roll follower, such that the center-line of the follower is ½ unit to the left of the vertical center-line, and the center of the roll is one unit above the shaft center, when in its lowest position.

Required. Lay out a plate cam that may be keyed to the shaft *A* and that will cause the follower *B* to rise 1½ units with uniformly accelerated and retarded motion while the cam rotates in a clockwise direction through 180°. The follower suddenly drops to its original position and remains at rest for the remainder of the rotation, 180°.

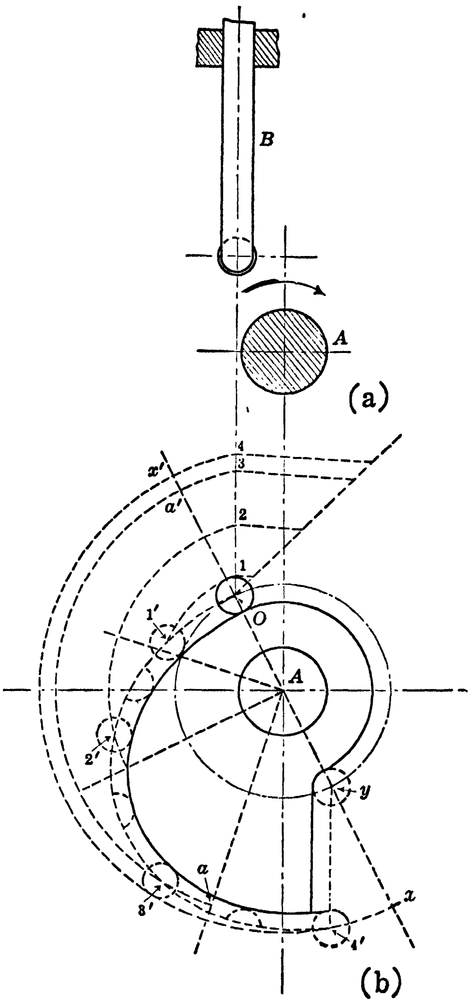


Fig. 127

Construction. First consider the center of the roll as a point follower, and draw the base circle with A as a center and AO as a radius, Fig. 127b. Lay off O–4 on the center-line of the follower equal to the required movement of the follower, and divide this distance into four parts successively proportional to 1, 3, 3, 1. Lay off the angle OAx equal to 180°, and divide this angle into four equal parts. With A as center and radius A–4, draw an arc cutting the fourth angle division line at x. The point x would be the position of the roll-center if the roll moved radially along the path AOx' through four time intervals. But since the follower path makes an angle $4Ox'$ with this radial line, the roll-center is offset a distance $x'4$ clockwise from AOx at the end of four time intervals. Therefore from x lay off $x4'$ equal to $x'4$. The point $4'$ will be at 4 when the cam has rotated through 180°.

Similarly, find the point *a*, and make the arc *a3′* equal arc *a′3*. Locate the points *1′* and *2′*. Draw a smooth curve through these points. This will give the portion of the theoretical curve required to raise the follower. The follower has a sudden drop to its original position; therefore draw a straight line from *4′* to *y*, where the radial line *Ax* cuts the base circle. The remainder of the curve will be the arc of a circle with a radius of *AO*. This completes the theoretical cam curve. Using the theoretical curve as a line of centers, draw the roll any number of times. The curve drawn tangent to these rolls will be the actual cam outline.

The general principles involved in the layout of a cam for an off-set follower are the same as those involved in the layout when the center-line of the follower passes through the shaft center. The main precaution to take is to make allowance for the angular difference between radial line *AO* and the follower path *O4*. This can be done in various ways, and the preceding method is only one means of obtaining the desired points on the cam curve. The rotation of the cam for an off-set follower should preferably be opposite to the direction of the off-set; for instance, if the follower is off-set to the right, the cam should rotate counter-clockwise. This will lessen the angle between the follower center-line and the force exerted between the cam and follower, thus reducing the component of the force normal to the follower center-line, and nearly eliminating the side-thrust against the follower guides.

A careful study of the preceding examples should make the design of point and roll-follower plate cams clear so that the student can lay out these cams without any difficulty.

133. Flat-Face Followers. The wear on the parts due to the use of pointed followers and the added cost of using roll followers may both be reduced by the use of a flat-face follower. The following example illustrates the layout for a flat-face reciprocating follower acting with a plate cam. Two conditions are possible, one where the face of the follower acting against the cam is perpendicular to the follower center-line, and the other where the face makes an angle other than 90° with the follower center-line. The example will consider the second case.

Given. A shaft *A*, Fig. 128a, and a flat-face follower, *B*, such that the center-line of the follower stem passes through the center of the shaft. The face of the follower makes an angle θ with a vertical center-line. The point *O*, the intersection of the face with the vertical center-line, is one unit above the shaft center when in its lowest position.

Required. Lay out a plate cam that will rotate with the shaft *A* and that will cause the follower *B* to rise $1\frac{1}{2}$ units with simple harmonic motion while the cam rotates in a counter-clockwise direction through 180°. The follower returns to its original position with the same motion while the cam rotates through the next 180°.

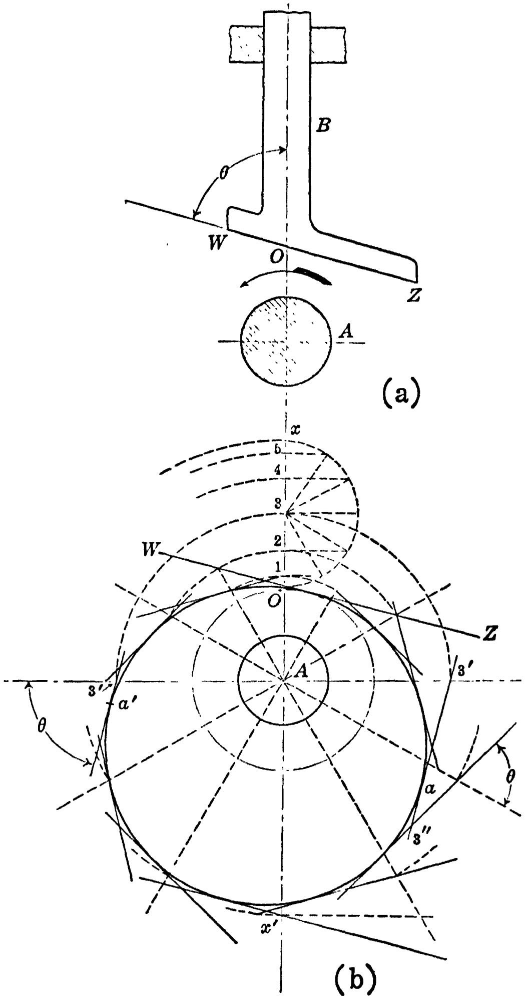


Fig. 128

Construction. Make *O–x*, Fig. 128b, equal to $1\frac{1}{2}$ units, the distance the follower is to rise, and divide *O–x* for harmonic motion, using any number of divisions (in this case six). Draw the base circle with *A* as center and tangent to the face. Divide the first 180° into six equal divisions. Take any point, as *3*, with radius *A3*, and rotate about *A* until the corresponding division line is crossed, giving the point *3′*. Draw a straight line *3′3″*, making angle *A3′3″*, equal to angle *AOZ*. This line represents the position of the face of the follower if it were revolved through three divisions to the right. All other points and lines are found in the same manner both for raising and lowering the follower. A smooth curve drawn tangent to the respective lines will be the outline of the cam.

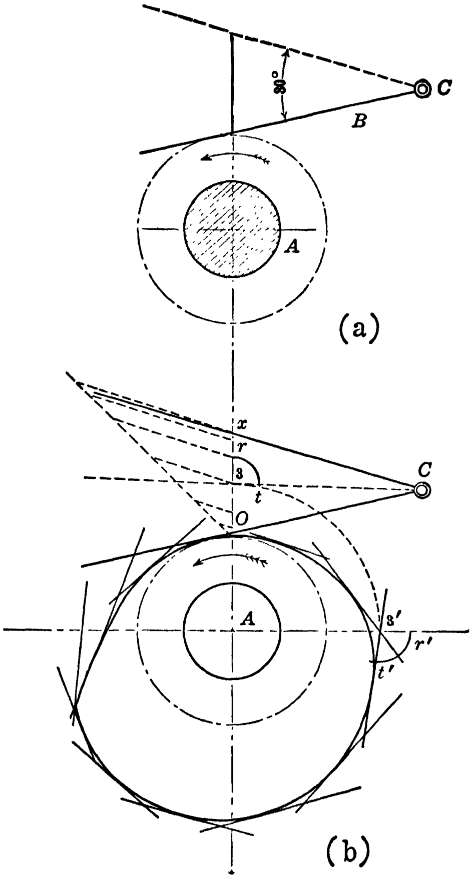


Fig. 129

To get the length of the face of the follower *WZ*, take the longest distance from the point of tangency of the curve with the line representing the follower, and the point this line was drawn through, such as *3′a*, and make *OZ* equal *3′a* plus ¼ inch or more. Now *OW* can be made equal to *OZ* for symmetry, or it can be found in the same way.

If the face had been perpendicular to the vertical center-line, the procedure would have been the same as in the preceding problem, the angle θ being made 90°.

134. Oscillating Followers. Consider a shaft *A*, Fig. 129a, rotating counter-clockwise, and a follower *B* pivoted at *C*. The point *C* is 2 units to the right and 1½ units above *A*. Diameter of base circle equals 2 units.

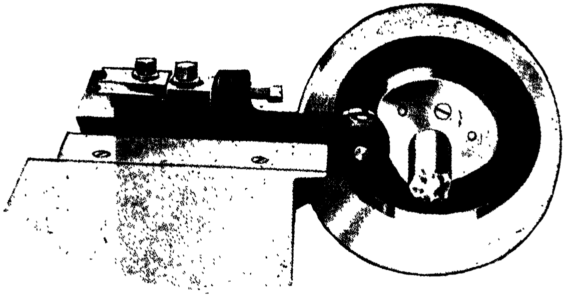
Required. Design a plate cam that will rotate with *A* and cause the follower *B* to oscillate through 30° with uniformly accelerated and retarded motion. *B* is raised during 180° rotation of the cam and lowered during the next 180°.

Construction. Divide the distance *O–x* into six divisions for uniformly accelerated and retarded motion. Divide 180° of the base circle into six equal parts, Fig. 129b. To lay out the cam, take any division of the follower, as *3*, and rotate this point about *A* cutting the corresponding division line at *3′*. Draw a line through *3′* so that angle *r′3′t′* equals angle *r3t*, thus locating the follower face for this position. Locate the follower in other positions in the same manner and then draw a smooth curve tangent to these lines. This curve will be the outline of the plate cam. The point at which this curve is tangent to one of these lines represents the point of contact between the follower and the cam, when each division line has rotated to the position *AOx*.

Oscillating followers may have either a roll, a flat face, or a point in contact with the cam.

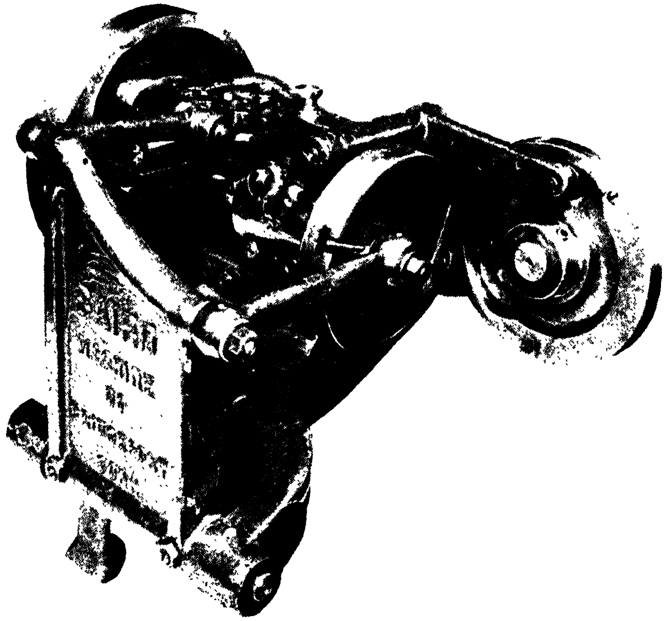
135. Positive-Motion Plate Cams. In many cases the follower may move in a horizontal line, or may be below the cam, so that gravity will not cause the follower to travel over the return stroke. It is then necessary to provide some device, such as a spring, to keep the follower against the cam surface. If the cam is rotating at a high speed, an exceedingly strong spring may be required to produce the acceleration necessary to keep the follower against the cam surface during the return stroke. Heavy springs are generally unsatisfactory, and some positive means of returning the follower is desirable. The method of designing several different forms of positive-motion cams is discussed in the following articles.

136. Face Cams. Figures 130 and 131 illustrate plate cams grooved to give positive motion. The layout is similar to those for roll followers described in the preceding articles, except that the curve must be laid out for both sides of the roll follower. This type of cam is the most common of the positive-motion cams and is frequently found on high-speed automatic machinery.



Baird Machine Co., Bridgeport, Conn.

FIG. 130. POSITIVE-MOTION CAM USED AS A TOOL-FEEDING DEVICE.



Baird Machine Co., Bridgeport, Conn.

FIG. 131. POSITIVE-MOTION CAMS ON A WIRE-FORMING MACHINE.

137. Constant-Diameter Cams. A constant-diameter cam is a plate cam in which the distance from edge to edge of the theoretical cam curve measured through the center of rotation remains constant. A constant-diameter cam is usually employed with two roll followers rigidly connected to give positive motion. The disadvantage of this type of cam is that the desired motion can be independently determined for only one-half revolution. The motion in the second half revolution is the reverse of the first, as the following example will indicate.

Given. The double roll follower and the cam shaft *A* as shown in Fig. 132.

Required. Design a constant diameter cam to raise the follower one unit with harmonic motion in 180°. The rotation is clockwise.

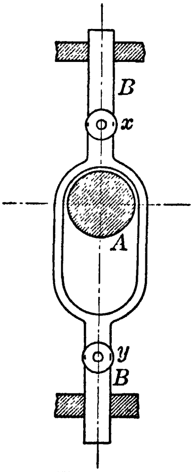


Fig. 132

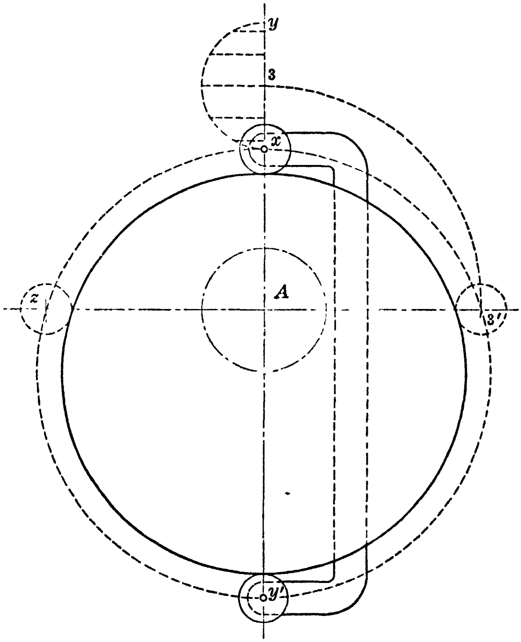


Fig. 133

Construction. Lay out the motion on the follower and proceed with the construction of the theoretical curve as described in § 131 for the first 180°. To find a point in the second 180°, Fig. 133, from point $3'$ on the theoretical curve just found, lay off $3'z$ equal to xy' on a diameter through A. Then z is a point on the theoretical curve. The actual or working curve is found by drawing a curve tangent to the rolls.

138. Constant-Breadth Cams. A constant-breadth cam is a plate cam in which the perpendicular distance between parallel tangents drawn on opposite sides of the cam remains constant. The predetermined motion of the constant-breadth cam is restricted to one-half revolution the same as with the constant-diameter cam. In this mechanism, two parallel flat-face followers are fixed to the same frame, and the construction is the same as that previously described for any flat-face follower for the first 180°. The second 180° is found by making the perpendicular distance between any two parallel tangents to the cam surface a constant equal to xy', as illustrated in Fig. 134, where $2'5'$ is equal to xy'.

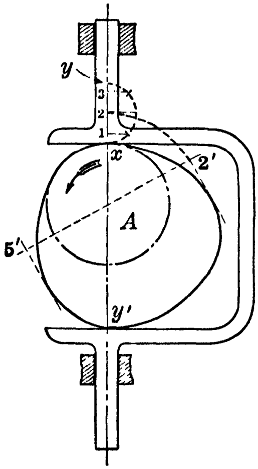


Fig. 134

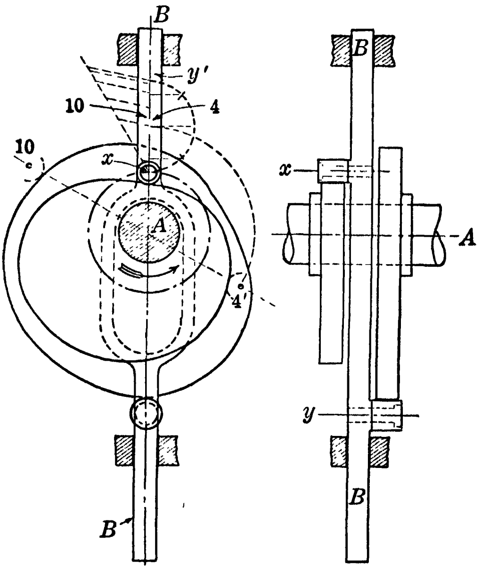


Fig. 135

139. Main-and-Return Cams. The main-and-return cam, consisting of two cams attached to the same shaft with the followers rigidly connected, gives a positive motion that does not limit the predetermined motion to one-half revolution. *Roll or flat-faced* followers may be used with this cam. The constant-diameter or constant-breadth principle is used in its design.

Given. A shaft *A*, turning counter-clockwise, and a double roll follower *B* as shown in Fig. 135.

Required. The outline of a plate cam that will cause the follower to be raised $1\frac{1}{2}$ units with uniformly accelerated and retarded motion in 240° and will lower the follower to its original position in the next 120° with harmonic motion.

Construction. To construct the main cam, lay out the required motion on the follower, Fig. 135, and find the points *1'*, *2'*, *3'*, *4'*, etc. on the theoretical curve as described in § 131. To find the points for the theoretical curve of the return cam, from the point *4'*, lay off *4'10* equal to *xy*, on a diameter through *A*. Point *10* is a point on the theoretical curve of the return cam. The actual working curves are found by drawing curves tangent to the rolls drawn with centers on the two theoretical curves.

140. Involute Cams. The involute of a circle is developed by a tracing point on a flexible cord as the cord is unwrapped from the circle as in Fig. 136.

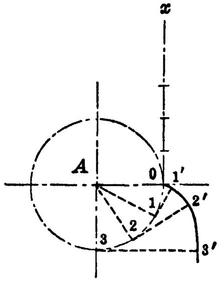


Fig. 136

This curve is often used for the outline of cams for stamping machines or machines for similar purposes. The center-line of the follower *ox*, in the figure, is usually tangent to the base circle, and in this case the follower is raised with uniform velocity. Such cams in stamping mills raise the follower, which then is allowed to drop to its original position by gravity. It is not uncommon to have two involute curves on the same cam so that the follower is raised twice in one revolution.

141. Cylindrical Cams. A cylindrical cam is formed if a groove is cut around a cylinder, of such shape as to give a definite motion to a follower which it actuates as the cylinder rotates. The grooved type is, by construction, a positive motion cam, and the follower may reciprocate or oscillate. A roll follower is generally used with this type of cam. The cam can be designed to give the follower any of the common motions previously referred to.

If one side of the groove on a cam is completely cut away so that there is only one side of the groove to act on the follower, it is often called an ***end cam.*** An end cam is not a positive-motion cam.

Sometimes the follower is actuated by straps fastened to the surface of the cylinder instead of by a groove in the cylinder. This type of cam is called a ***strap cam*** or ***drum cam.*** This type of cam is adjustable, since the straps may be changed to give different motions to a follower without changing other conditions in the cam mechanism, and therefore is extensively used on automatic screw machines. This type may or may not be a positive motion cam.

A cylindrical cam can best be laid out by making an elevation and development of both the top and bottom of the groove. The following example will illustrate this method.

Given. A cylinder 3 units in diameter and 3 units long, turning clockwise as seen from the right, and a roll follower located above the cylinder, moving parallel to the axis of the cam. The roll is $\frac{5}{8}$ of a unit in diameter at the large end and of such shape as to give pure rolling contact. The groove is $\frac{5}{8}$ of a unit deep.

Required. Lay out the elevation and development of both the top and bottom of a groove that will cause the roller to move $1\frac{1}{2}$ units to the left with harmonic motion in $\frac{3}{8}$ turn of the cam, stand still $\frac{1}{8}$ turn, and return to its original position in the remaining $\frac{1}{2}$ turn with harmonic motion.

Construction. Draw the cylinder and the left end view, Fig. 137, the dotted circle representing the bottom of the groove. Divide this circumference into the three main parts, 135°, 45°, and 180°, for the three different motions. Lay off the diameter of the top of the roll on the surface of the cam at *ab*. Draw the center-line *rs* bisecting *ab*. Draw lines from the points *a* and *b* to the center of the cylinder *z*, which will form the sides of a roll to give pure rolling contact. The line *cd* is the bottom of the groove, and *abdc* is the true size of the groove in this position.

To develop the outside of the groove, draw a line xy, Fig. 138, equal to the circumference of the cylinder. Divide xy into three spaces, $x4$, 4–5, and $5y$. Make xl equal to $1\frac{1}{2}$ units, the total movement of the follower. Divide xl for harmonic motion, using six divisions, and divide $x4$ into the same number of equal parts. The intersection of the corresponding vertical and horizontal lines will give points on the center-line $lc4$. Since the cam is at rest for the next $\frac{1}{8}$ turn, its path on the developed surface will be a straight line along xy, from 4 to 5. Lay off the curve $5fg$ in the same manner for the return stroke of the follower. Draw circles representing the large diameter of the roll with centers on $lc45fg$. Draw smooth curves tangent to each side of the circles,

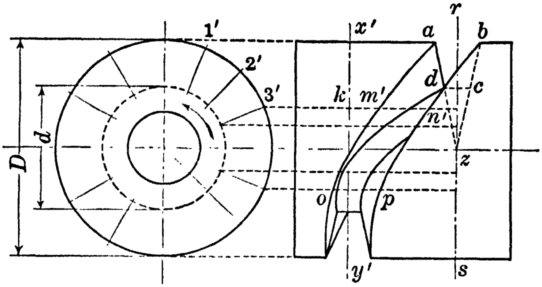


FIG. 137

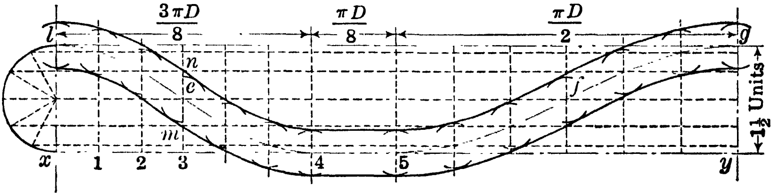


FIG. 138

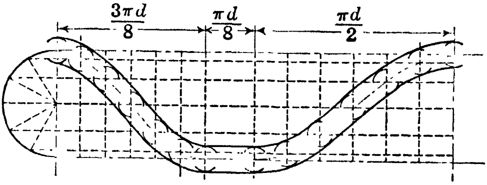


FIG. 139

completing the development of the top of the groove for the circumference of the cylinder. In like manner the development of the bottom of the groove is obtained, as shown in Fig. 139.

Transfer the developed curves to the cam cylinder, completing the drawing of the cam.

It is not necessary to draw a complete development and elevation of both the top and bottom of the groove. For most work the development of the top of the groove and a drawing showing the shape of the groove are sufficient. The developments are used for studying the motion of the follower. The center-lines of the grooves may be plotted directly on the cylinder. Figure 140 shows the development of the curve directly on the cylinder for the same conditions used in Fig. 137.

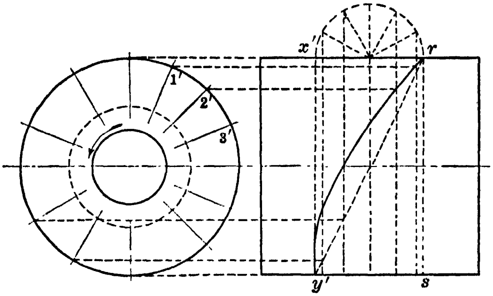


Fig. 140

A cylindrical cam may operate an oscillating follower instead of a reciprocating follower, and the construction will have to be modified to allow for the swing of the follower. The following example will illustrate the construction for this type of follower.

Given. A cylinder and an oscillating roll-follower, as indicated in Fig. 141. The follower is to move from A to A' with uniform velocity in 120° of rotation of the cam, rest 60°, move from A' to A with uniform velocity in 120°, and rest for 60°. The follower is to oscillate through equal angles on both sides of the center-line BC. The arm AB is of such length that it moves an equal distance above and below the horizontal center-line MN. The cam turns clockwise as seen from the right, and the roll is of such shape as to give approximately pure rolling contact.

Required. Draw the layout of the center-line for the complete development of the top of the groove and the development of the center-line of the groove on the cylinder.

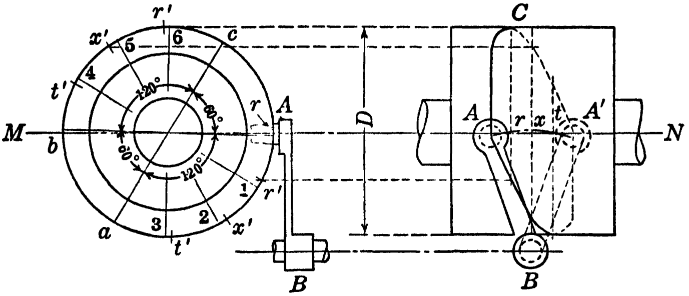


Fig. 141

Construction. Divide xy, Fig. 142, into four parts to correspond to the different periods of motion and rest. The length of xy corresponds to the outside circumference of the cam, and AA' equals the travel of the follower. Divide AA', Fig. 141, into four equal parts for uniform motion, and divide xa into the same number of equal parts. With AB as a radius and the center along xy, draw arcs through x, *1*, *2*, *3*, etc. The points of intersection between horizontal lines drawn through A, r, x, etc. with the corresponding arcs drawn through *1*, *2*, *3*, etc. will give points on the center-line for the groove. The complete center-line is $AEFGH$. The rollers can be drawn in with centers on this line, and the curves for both the top and bottom of the groove developed as in the preceding example if it is desired. The center-line may be laid out directly on the cylinder as shown in Fig. 141, where the construction for two points is indicated.

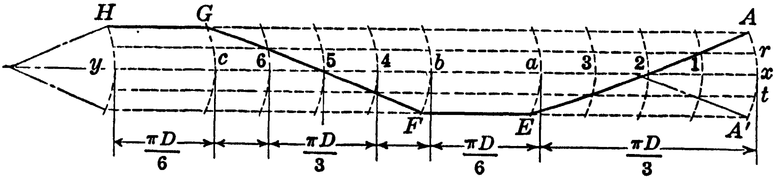
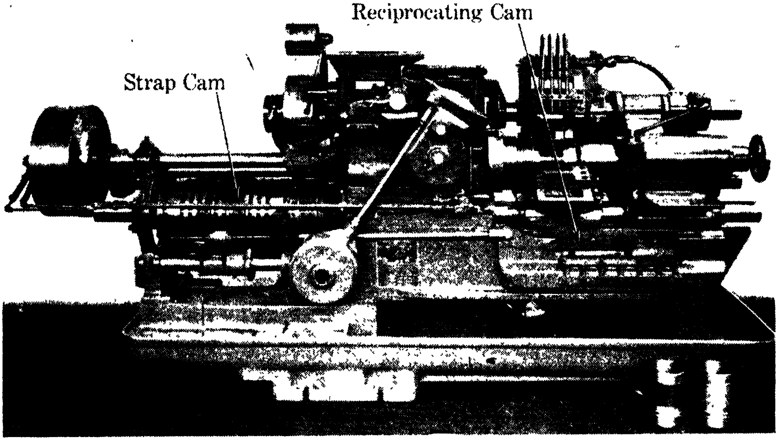


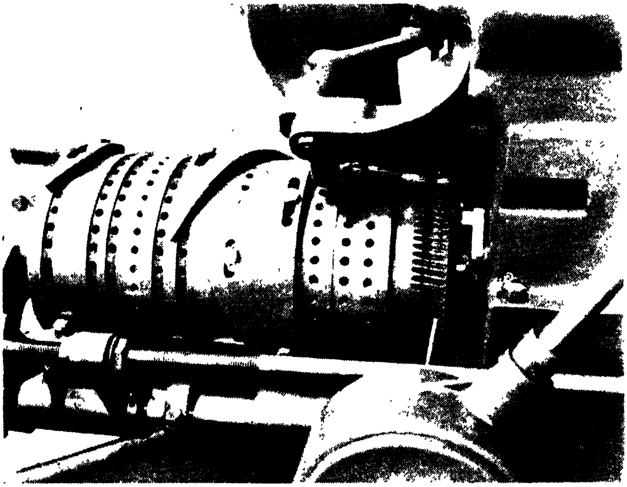
Fig. 142

142. Strap Cams. Strap cams are illustrated in Figs. 143 and 144. The straps are shaped so as to give the desired motion and are fastened to the drum. They may or may not be adjustable.



Jones and Lamson Machine Co.

FIG. 143. LATHE EQUIPPED WITH A STRAP CAM AND A RECIPROCATING CAM.



Jones and Lamson Machine Co.

FIG. 144. STRAP CAM WITH ADJUSTABLE STRAPS.

143. Conical and Spherical Cams. The cams designed in preceding articles for operating reciprocating followers have moved the follower either parallel or perpendicular to the axis about which the cam rotates. If it is desired to operate a reciprocating follower which makes an angle with the axis about which the cam rotates, a conical cam may be employed. This cam is designed so that a groove cut in a cone gives the desired motion to the follower.

The spherical cam is formed by cutting a groove in a sphere to give the required motion to the follower. This type of cam is employed to operate a swinging follower whose center is at the center of the sphere.

144. Oscillating Cams. Oscillating cams are used where complete rotation of the cam is undesirable, Fig. 145. The theory of this type of cam and the method of layout of the cam curves are the same as already described for rotating cams. However, in many cases the cam is oscillated by a link and crank mechanism and does not oscillate through equal angles in equal time units so that the space-time divisions on the base circle will not be equal. The divisions can be obtained, however, from the known movement of the driving crank and link.

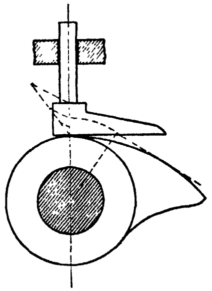


FIG. 145

Such cams are used for opening the poppet-valves of large steam and gas engines. The links C and F in Fig. 106 are cams operating the valves of a large engine. If sliding between the cam and follower surfaces is undesirable, the follower may be so shaped as to give rolling contact. The outline of such rolling surfaces is discussed in Chapter 12.

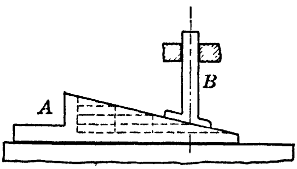


FIG. 146

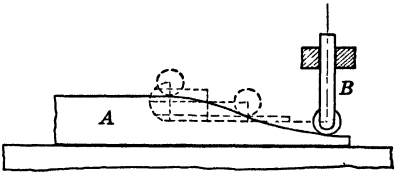


FIG. 147

145. Reciprocating Cams. When the radius of oscillation of the cam becomes infinitein length, the cam becomes a reciprocating cam. The motion of the follower is usually at right angles to the motion of the cam. Layouts for different types of followers are shown in Figs. 146, 147, and 148. The construction is indicated in the figures and does not differ from other forms of cams.

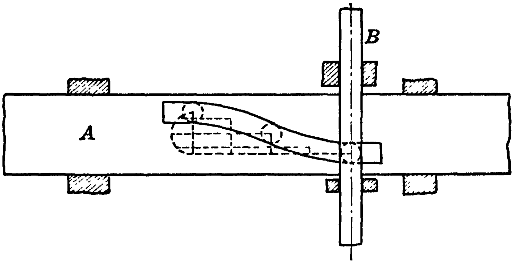


FIG. 148

This type of cam is used to operate valves, reducing-motions, etc. Figure 143 shows a reciprocating cam used on a lathe to feed the cutting tools into the work.

146. Velocity of Follower. It is sometimes necessary to know the linear or angular velocity of the follower for some point in the cycle or to know the velocity curve for the entire cycle. There are two conditions to be considered, one where the follower is reciprocating and one where the follower is oscillating.

147. Velocity of Reciprocating Followers. Reciprocating followers may be in the form of a point follower, roll follower, or flat-face follower; but the method of finding the velocity at any point is the same, regardless of the form. The following example will illustrate the method of finding the velocity-space diagram of a reciprocating point follower.

GIVEN. A plate cam rotating at 100 r.p.m. about the center *A* as shown in Fig. 149. The cam raises the follower *B*, 3 inches in 180° and lowers the follower to its original position in the next 180°.

REQUIRED. Draw the velocity-space diagram of the follower.

Construction. Draw the line xy equal to the travel of the follower. The angle during which the follower rises is divided into six equal parts. Lay off points on xy to represent the displacement of the follower for each division of rotation of the cam. The velocity of any point m about A can be calculated. Let the vector mp represent this velocity to scale. Draw mn tangent to the cam curve at m and pn parallel to Am. Then pn is the linear velocity of the follower, and mn is the relative velocity or velocity of slip between the follower and the cam at the instant when the follower point is in contact with m during the cam revolution. Plot pn to the right of xy as $p'n'$. The linear velocities for other positions of the follower are found in the same manner. The velocities of the follower while lowering are plotted to the left of xy. When the velocity-space diagram, $xn'y$, is found, the acceleration-diagram, $cp'e$, can be found by the method described in § 90.

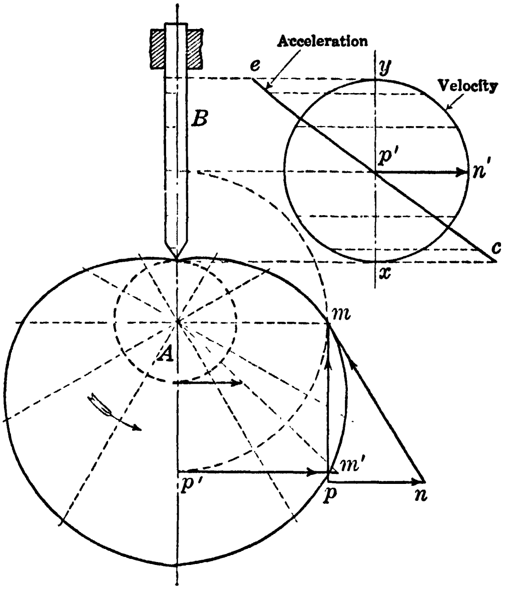


Fig. 149

The following method may be used to determine the velocity of a reciprocating follower instead of the one given above.

In Fig. 150, the cam and follower may be considered as a four-bar linkage with some of the links imaginary. *DCE* is part of the surface outline of a cam rotating clockwise about the center *A*. A line drawn through *B* and *C*, the point of contact of the roll and the cam, is normal to the cam curve. This line is the imaginary connecting-rod. The rod *BF* and its guide are equivalent to a crank perpendicular to *BF*, with its center *G* at infinity. The second crank is any line from *A* to the connecting rod *BC* as *AH*. If the velocity of any point on the cam is known, or if the angular velocity is known, the velocity of point *H* can be determined; and by any method of transferring velocities in linkages, the velocity of *B* can be found. In the figure, this was done when *Bm* was drawn parallel and equal to the velocity of *H* and *mb* was drawn perpendicular to *BC*, cutting *BF* at *b*. The vector *Bb* represents the velocity of *B*.

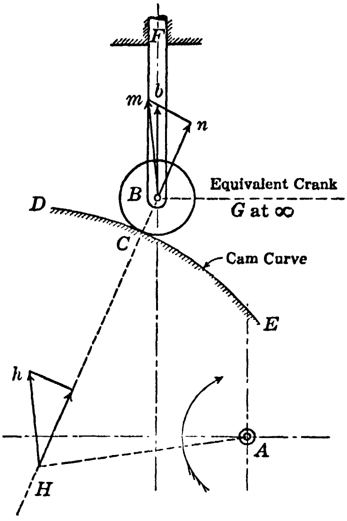


FIG. 150

If a flat-faced follower had been used, the construction would have been the same. In this case, the imaginary connecting-rod is perpendicular to the follower face at the point of tangency.

148. Oscillating Follower. An oscillating follower may have a point, roll, or a flat face working against the cam. The method of finding the linear or angular velocity of the follower is the same, regardless of the type used.

GIVEN. The plate cam *A* oscillating the flat-face follower *OB* as shown in Fig. 151.

REQUIRED. Find the ratio of the angular velocity of the cam to that of the follower when the cam has rotated through 90°.

CONSTRUCTION. When the cam has rotated through 90°, the follower will be in the position *OB′*, and the cam and the follower will be in contact at x (§ 134). Draw a perpendicular to the face of the follower at x', cutting the line of centers OA at m. Then

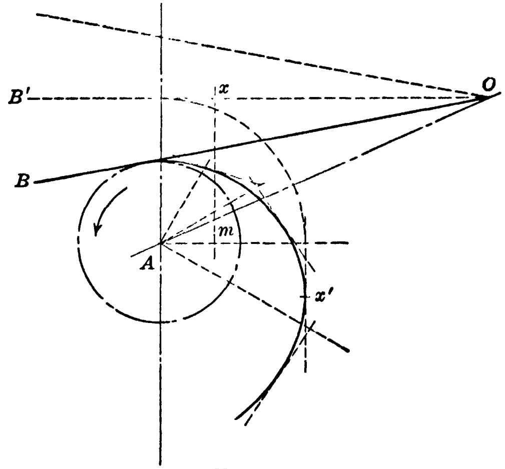


FIG. 151

$$\frac{\text{angular velocity of cam}}{\text{angular velocity of follower}} = \frac{mo}{mA}.$$

For proof of this construction, see § 55. The ratio of the angular velocities for any other position can be found in a like manner.

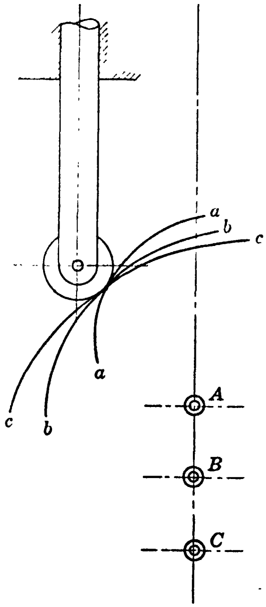


FIG. 152

149. Size of Cam. In all the layouts for cams previously discussed, the diameter of the base-circle was arbitrarily assumed, and this assumption fixed the size of the cam. It is clear that, if the diameter of the base circle is changed, the outline of the cam will be changed. In Fig. 152, the cam outline has been drawn for the same motion of the follower but with different centers of rotation for the cam, as A, B, and C and the cam outline for the same angular movement of the cams being aa, *bb*, and *cc*, respectively. The smaller the diameter of the base circle, the steeper is the cam outline, the greater the side thrust on the follower, and the greater the friction loss in transmitting power. In order to keep this friction loss within reasonable limits and to assure easy operating conditions, the angle between the center line of the follower and the line through the center of the roll normal to the cam curve should not exceed 45°. The minimum size of the cam, therefore, will be determined by the above angle. The minimum size may also be limited by the cam shaft or other structural details. The maximum size of the cam is limited only by the structural details of the machine on which the cam is to be used.

150. Size of Roll. The radius of the roll should be less than the smallest radius of curvature on the concave portion of the cam outline. If the radius of the roll is greater than the radius of the cam curve, the desired follower-motion will not be obtained. The radius of the roll may be made equal to the smallest radius of the cam curve, but this is undesirable because of the sharp corner developed on the theoretical curve, or pitch line, under such conditions, which indicates a sudden change in the follower velocity, which in turn produces unnecessary shock.

PROBLEMS

1. A body is to rise in a straight vertical line, a distance of 2 in. in 10 sec. Plot a space-time diagram for each of the following cases. Ordinates to be the distance moved to full scale. Abscissae to be time elapsed to the scale 1 in. equals 4 sec.

(a) Rise with uniform velocity.
(b) Rise with harmonic motion.
(c) Rise with uniformly accelerated motion.
(d) Rise 1 in. with uniform acceleration and the remaining 1 in. with uniformly retarded motion.

2. The ram of a certain stamp mill is lifted 6 in. in 10 sec. by a cam and then allowed to drop by the action of gravity alone. During the upward movement, it is uniformly accelerated for 3 sec., rising 2 in.; has uniform velocity for 5 sec., rising 2 in.; is uniformly decelerated during the remaining 2 seconds, coming to rest at the top position. Graphically divide the 6-in. line to indicate the position of the stamp at the end of each ½ sec.

3. A body is to rise and fall according to the data given. Plot a space-time diagram, plotting space traversed as ordinates to the scale 1 in. equals 2 in., and the time elapsed as abscissae to the scale 1 in. equals 8 sec.

Rise 1 in. in 4 sec. with uniform acceleration; rise 1 in. in 4 sec. with uniform retardation; rest 6 sec.; fall 1¼ in. in 6 sec. with harmonic motion; rise ½ in. in 3 sec. with uniform velocity; fall 1¼ in. at once; rest 4 sec.

4. A plate cam is to move a pointed follower as follows: rise 1 in. with uniform velocity during 180° revolution of the cam; rest 90°; drop 1 in. with harmonic motion in 90°. The path of the follower passes through the center of rotation of the cam. The lowest position of the follower is 1¼ in. above the center of rotation. The cam rotates counter-clockwise. Lay out the cam outline.

5. The valve cam used on a gas engine consists of two circular arcs whose centers are 1½ in. apart, joined together by tangent lines. The circle arcs have ¾- and ¼-in. radii respectively, and the cam rotates about the center of the larger arc.

(a) Plot the lift of the follower against the valve rotation in degrees, using a point follower whose path (continued) passes through the center of rotation. Scale of lift, full size; scale of rotation, 1 in. = 120°.

(b) Same as (a), except substituting on the follower a flat face perpendicular to the direction of lift.

6. In the laboratories and shops on the campus, or in nearby plants, locate at least ten uses of cams. Sketch the cam and describe the function for which it is used.

7. A plate cam revolving clockwise is to move a pointed follower as follows: rise 1¼ in. with uniform acceleration during 90° revolution of the cam; rise ¾ in. with uniform retardation in 90°; rest 30°; fall 2 in. with uniform velocity in 150°. The path of the follower passes through the center of rotation of the cam and the lowest position of the follower is 1½ in. above the center. Lay out the cam outline.

8. Draw a cam with the data given in Problem 4 but move the path of the follower ½ in. to the right of the center of rotation of the cam.

9. Draw a cam with the data given in Problem 7 but move the path of the follower ¾ in. to the left of the center of rotation of the cam.

10. A plate cam with a roll follower ½ in. in diameter is to rotate counter-clockwise. The lowest position of the center of the roll follower is to be 1½ in. above and ½ in. to the right of the center of

rotation of the cam. The path of the follower is to be in a vertical line, and the motion of the follower is to be as follows: rise 1 in. with harmonic motion in 90° rotation of the cam; rest 90°; rise 1 in. with uniform motion in 60°; drop 2 in. at once; rest 120°. Lay out the cam outline.

11. A plate cam rotating clockwise has a $\frac{3}{8}$-in. roll follower moving in a vertical line $\frac{3}{4}$ in. to the left of the center of rotation of the cam. The follower is to move as follows: rest 30°; rise $\frac{3}{4}$ in. in 60° with uniform motion; fall $\frac{1}{2}$ in. in 30° with uniform motion; rest 60°; rise 1 in. with harmonic motion in 90°; fall 1 in. at once; rise 1 in. in 90° with uniform motion; fall $1\frac{1}{4}$ in. at once. The lowest position of the center of the roll follower is 1 in. above the center of rotation of the cam. Determine the cam outline.

12. A cam with a roll follower $\frac{1}{2}$ in. in diameter is to lift its follower $1\frac{1}{2}$ in. with uniformly accelerated and retarded motion in 180°, rest 60° and lower, with sudden drop, $1\frac{1}{2}$ in. The lowest position of the center of the roll is $\frac{1}{2}$ in. to the right and $1\frac{1}{4}$ in. above the center of rotation of the cam. Lay out the cam for counter-clockwise rotation and indicate and describe the construction.

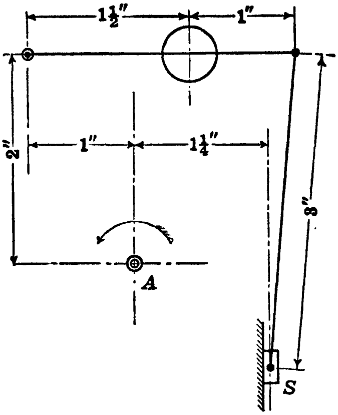
13. Design a cam, using the data in Problem 10 except that the path of the follower is to be along the arc of a circle whose center is $1\frac{1}{2}$ in. above and $2\frac{1}{2}$ in. to the right of the center of rotation of the cam.

14. A plate cam rotating clockwise has a flat-footed follower whose face makes an angle of 90° with the direction of its motion. The smallest radius of the cam is to be $1\frac{1}{2}$ in. The follower is to rise with uniform velocity a distance of $1\frac{1}{2}$ in. during 180° rotation, and fall with harmonic motion a distance of $1\frac{1}{2}$ in. in 180°. Determine the cam outline.

15. A plate cam rotating counter-clockwise moves a flat-footed follower whose center of oscillation is 4 in. to the right on a horizontal line through the center of rotation of the cam. The face of the follower is radial with its center of oscillation. The smallest radius of the cam is to be 1 in., and its largest radius 2 in. The follower is to rise the full distance during 90° rotation of the cam with uniform motion, rest 90°, and return to its original position in 180° with uniform motion. Determine the outline of the cam. Indicate and describe the method of construction.

16. A crank-and-crosshead mechanism has the following dimensions: crank $1\frac{1}{2}$ in. long and connecting-rod 5 in. long. Design a plate cam that will operate a flat-face follower and duplicate the motion of the crosshead. The smallest radius of the cam is to be $1\frac{1}{2}$ in. Indicate and describe the method of construction.

17. Design a plate cam to move the slide *S* in the figure with the following motion: rise 1 in. with harmonic motion in 120°; rest 60°; fall ½ in. with uniform velocity in 90°; fall ½ in. at once; rest 90°.



Problem 17

18. Lay out a cam revolving clockwise to operate two rollers, each ½ in. in diameter and located 3 in. apart on a rigid frame. The rolls move on a horizontal line passing through the center of rotation of the cam. The rolls move 2 in. to the right with harmonic motion in 180° rotation of the cam. What kind of motion will the rolls have during the return stroke?

19. Two rollers, ½ in. in diameter and 4 in. apart, are attached to the same frame and are in contact with the same plate cam. The center of rotation of the cam is on the line joining the roll centers. Find the outline of the cam if the frame is to be raised 1½ in. in ½ revolution with harmonic motion.

20. The outline of one-half of a cam consists of a semicircle 2 in. in diameter rotating about a center ¾ in. from the disc center and on a line bisecting the arc. The follower is actuated by two ½-in. rollers 3 in. apart between centers. Lay out the other half of the cam outline.

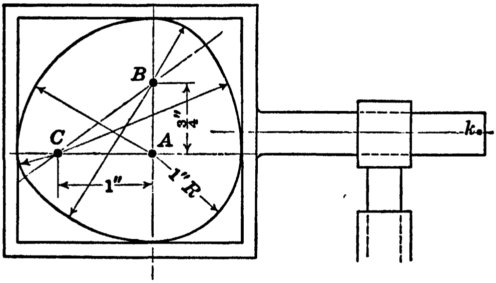
21. Two rollers 5 in. apart on a horizontal line are attached to a rigid frame. Each roll is ⅝ in. in diameter. The rollers are to move 2 in. to the right at uniform velocity in 120° rotation of the cam, move 1 in. to the right with uniform retardation in 30°, and rest 30°. Lay out a cam with the center of rotation on the line of centers of the rolls to give this motion. What will be the motion of the rollers during the return stroke?

22. Substitute flat vertical surfaces for the rollers in Problem 21 and lay out a cam to give the motion specified. To what class does this cam belong?

23. Lay out a constant-breadth cam to move its follower 2 in. up with harmonic motion. The smallest radius of the cam is to be 1 in.

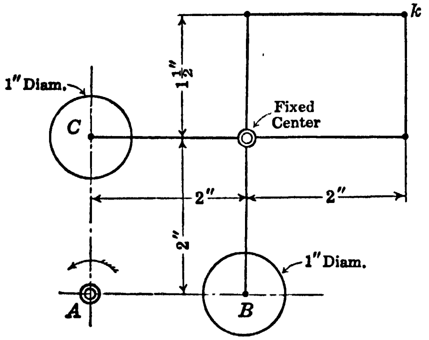
24. The outline of a cam is the involute of a circle of 4-in. diameter, and the flat-face follower moves in a straight line passing 3 in. from the cam center. Does the follower move with uniform or non-uniform motion? Prove your statement.

25. In the figure the cam rotates clockwise. Plot the motion of the point k, for one complete revolution of the cam.



PROBLEM 25

26. In the figure for Problem 25, change the angle BAC to 120° and plot the motion of the point k.



PROBLEM 27

27. In the figure, the point k is to move up ½ in. in a vertical line with uniform velocity during 90° revolution of the cam, move horizontally to the right ½ in. with uniform velocity in 90°, and return in a straight line to its original position with uniform velocity in 180°. Lay out two plate cams attached to the shaft A, to operate the rollers B and C so the point k will have this motion.

28. What type of cam is generally used in the following machines: automobile engine; Diesel engine; Poppet valve engine; turret lathes; mechanical-feed lubricating-pumps?

29. Make a sketch of three positive-motion cams; explain their construction and operation, and name a place where each is used.

30. Find an application (manufacturer's catalogs or other convenient place) of a plate cam, a cylindrical cam, a positive-motion cam, an oscillating cam, and a reciprocating cam.

31. A cylindrical cam, 4 in. outside diameter, turns left-handed as seen from the right. The follower is a roller which is to move parallel to the axis of rotation of the cam. The large end of the roller is to be ½ in. in diameter, and the roller is to be of such form as will give pure rolling contact. The groove in the cam is to be ½ in. deep.

Draw the development for both the top and bottom of the groove. Draw an elevation of the cam showing the top of the groove only. The follower motion is to be as follows: move 1 in. to the right with uniform velocity during 90° rotation of the cam; rest 90°; move to the left $1\frac{1}{2}$ in. with harmonic motion in 90°; move $\frac{1}{2}$ in. to the right with uniformly accelerated motion in 90°.

32. A plate cam rotates at 100 r.p.m., and raises a roll follower 6 in. with uniformly accelerated motion in $\frac{1}{2}$ revolution of the cam, and lowers the follower to its original position with uniformly decelerated motion during the remaining $\frac{1}{2}$ revolution.

(a) What is the maximum velocity of the follower during the upward motion? During the lowering?
(b) Would the follower stop at the end of its 6-in. path? Give reasons for your answer.
(c) Do you think a change in the design of the cam desirable? If so, what change would you make?

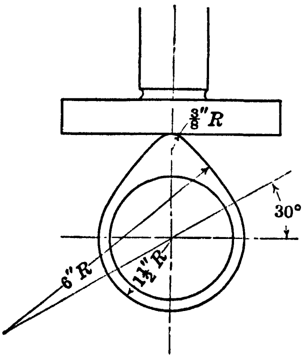
33. A cam consists of a circular disc 3 in. in diameter rotating at 10 r.p.m. counter-clockwise, about an axis 1 in. from the center of the disc.

(a) A point follower moving on a vertical line passing through the center of rotation is used. Determine the velocity of the follower when the center of the disc is horizontally to the right of the center of rotation.
(b) Same as (a) using a roll follower 1 in. in diameter.
(c) Same as (a) using a flat face perpendicular to the direction of motion.
(d) In (c) find the rubbing or sliding velocity between the cam and the follower face.

34. The cam shown in the figure rotates at 200 r.p.m. Determine the maximum velocity of the follower and the position of the cam at this time.

35. In Problem 34, substitute a 1-in. roller for the flat follower and solve.

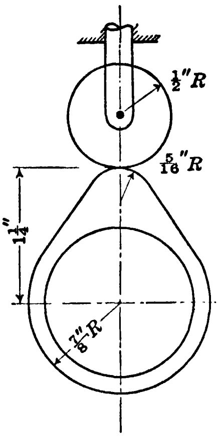
36. In the cam shown in Problem 34, determine the acceleration of the follower when the cam has turned 30°, 45°, and 60°.



Problem 34

37. In the cam shown in Problem 34, substitute a $\frac{3}{4}$-in. roller for the flat follower and determine the acceleration of the follower when the cam has turned 30°, 45°, and 60°.

38. In Problem 33, determine the thrust angle for each of the followers used.



PROBLEM 39

39. The cam of an internal combustion engine is shown in the figure. The cam turns at 500 r.p.m.

(a) Plot a velocity-space diagram for the follower. Indicate the scales used for the coordinates.

(b) Plot an acceleration-space diagram for the follower. Indicate the scales used for the coordinates.

40. The cam shown in the figure for Problem 34 turns at 300 r.p.m.

(a) Plot the velocity-space diagram for the follower, indicating the scales used for the coordinates.

(b) Plot the acceleration-space diagram for the follower, indicating the scales used for the coordinates.

41. Using the data of Problem 39, plot the velocity-time diagram and the acceleration-time diagram, assuming the cam to make 60 revolutions per min. Scale of time 1 in. = $\frac{1}{4}$ sec.; of velocity 1 in. = 1 in. per sec.; of acceleration 1 in. = 10 in. per sec.

42. Design a main-and-return cam. The main cam is to have a base circle 2 in. in diameter. The follower is to rise $1\frac{1}{2}$ in. with uniform motion in 180° and return with harmonic motion in 180°. The rolls on the followers are $\frac{5}{8}$ in. in diameter and are spaced $2\frac{1}{2}$ in. on centers.

43. Make a drawing showing the effect of the radius of the roll on the cam outline for the three conditions mentioned in § 150.

CHAPTER XI

FLEXIBLE TRANSMISSION

151. Discussion. In the transmission of motion or power from one rotating shaft to another, it is not always desirable or economical to use links or gears; so ***flexible connectors,*** such as ***belts, ropes,*** or ***chains*** are used. Theoretically, power or motion may be transmitted from one rotating shaft to another by either links, gears, or some form of flexible connector. Practically, the requirements of a given job will eliminate either one or two of the above methods on account of the consideration of costs or of Machine Design.

Belts are made of leather, fabric, rubber impregnated fabric, and steel; and they are used generally in transmitting power and in different types of conveyors.

Ropes are made of cotton, hemp, leather, wire, and similar materials, and are used in transmitting power and in hoisting.

Chains are classified according to type, such as block, roller, silent, etc., and are used in transmitting power, in hoisting, and in various types of conveyors and elevators.

Belts and ropes depend upon friction to transmit power, while chains do not; therefore belts and ropes are considered as ***friction drives,*** and chains as ***positive drives.***

152. Speed Ratios for Belt Pulleys. In Fig. 153, pulley A rotates at N_a r.p.m., and a point on its circumference has a linear speed of $\pi D_a N_a$ feet per minute. When a belt is passed over pulley A and extended around pulley B, as shown, it has a linear speed approximately the same as that of the surface of the pulley A or $\pi D_a N_a$. The surface of the pulley B will then have approximately the same linear speed as that of the belt; therefore

$$\pi D_a N_a = \pi D_b N_b,$$

whence

$$\frac{D_a}{D_b} = \frac{N_b}{N_a}. \tag{62}$$

The angular speed ratio of two shafts, with a belt drive, is inversely proportional to the diameters of the pulleys.

The above statement is theoretically true, but, practically, errors are introduced because of the thickness of the belt, the slippage of the belt on the pulleys, and the creep of the belt.

If a flat belt of known width and thickness, with the two ends fastened together, is stretched over two pulleys, as in Fig. 153, the following conditions are apparent. The surface of the belt next to the pulleys is shorter than before, and the outside surface is longer than before. Therefore the inside surface of the belt has been compressed, and the outside surface is under tension. It follows, then, that there is a surface between these two surfaces that has been neither compressed nor stretched, called the neutral surface, which is located about one-third of the thickness of the belt from the inner side. The speed of the belt may be calculated from this neutral element. However, since the thickness of the belt is usually small in comparison to the diameter of the pulley, it need not be considered.

The slippage depends on the type of installation, power to be transmitted, belt material, care, belt speed, etc., and cannot be calculated for a proposed installation. By experiment, the value of slippage has been found to range from 1 to 2½ per cent for good installations.

A condition, known as ***creep,*** results in a loss in speed due to the belt's being stretched a greater amount on the high tension side than on the low tension side when under load.

Errors due to the belt thickness and the slip of the belt tend to neutralize each other. In the subject of mechanism, such losses will not be considered. Speed calculations will be based on pulley diameters only.

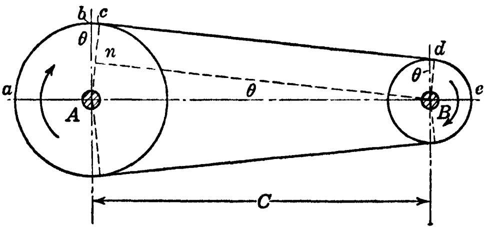


FIG. 153

153. Belt Drives. Belts can be used to connect all possible combinations of shafting, that is, shafts which are parallel, intersecting, and neither parallel nor intersecting.

Parallel shafts. Figure 153 shows a shaft, A, parallel to shaft B, the distance between centers being C. The shaft A rotates at N_a r.p.m. Therefore the diameter D_b of the pulley on B is

$$D_b = \frac{N_a}{N_b} \times D_a.$$

If the shaft A rotates clockwise, the shaft B can be made to rotate either clockwise or counter-clockwise, according to the way the belt is placed on the pulleys. If the pulleys rotate in the same direction, the belt is called an ***open belt,*** Fig. 153; and if the pulleys rotate in the opposite direction, the belt is called a ***crossed belt,*** Fig. 154.

With the belts as shown in Figs. 153 and 154, the angular velocity of B may be constant or variable, depending on the angular velocity of A; however, it is usually constant.

154. Length of Belts. It is necessary to calculate the length of belts to determine the amount of belt material needed for a new installation, to determine step and cone pulley dimensions, etc.

Open belts. In Fig. 153, the diameter of the pulley on the shaft A is D_a, and the diameter of the pulley on the shaft B is D_b. The distance between shafts is C. To derive the exact formula for L, the length of the belt, we write first

$$L = 2(ab + bc + cd + dc). \tag{63}$$

Drawing Bn parallel to dc, we find

$$ab = \frac{\pi D_a}{4}, \qquad cd = Bn = C \cos \theta,$$

$$bc = \frac{D_a \theta}{2}, \qquad dc = \frac{\pi D_b}{4} - \frac{D_b \theta}{2}.$$

It follows that

$$L = 2\left(\frac{\pi D_a}{4} + \frac{D_a \theta}{2} + C \cos \theta + \frac{\pi D_b}{4} - \frac{D_b \theta}{2}\right) \tag{64}$$

$$(65) \qquad = \frac{\pi D_a}{2} + D_a\theta + 2C\cos\theta + \frac{\pi D_b}{2} - D_b\theta,$$

or

$$(66) \qquad L = \frac{\pi}{2}(D_a + D_b) + \theta(D_a - D_b) + 2C\cos\theta.$$

We know that

$$\sin\theta = \frac{An}{AB} = \frac{D_a - D_b}{2C}, \qquad \cos\theta = \sqrt{1 - \frac{(D_a - D_b)^2}{4C^2}}.$$

When θ is small, θ in radians = $\sin\theta$ very nearly; hence we may write

$$(67) \quad L = \frac{\pi}{2}(D_a + D_b) + \frac{(D_a - D_b)^2}{2C} + 2C\sqrt{1 - \frac{(D_a - D_b)^2}{4C^2}}.$$

Expanding the term under the radical by the Binomial Theorem, and dropping all terms having a greater power of C than C^2 in the denominator, we find

$$(68) \quad L = \frac{\pi}{2}(D_a + D_b) + \frac{(D_a - D_b)^2}{2C} + 2C\left(1 - \frac{(D_a - D_b)^2}{8C^2}\right), \text{etc.}$$

$$(69) \quad L = \frac{\pi}{2}(D_a + D_b) + 2C + \frac{(D_a - D_b)^2}{4C}, \text{ nearly.}$$

Equation (66) is the exact formula for calculating the length of open belts, and equation (69) is modified, but it is nearly exact and may be used in belt calculations.

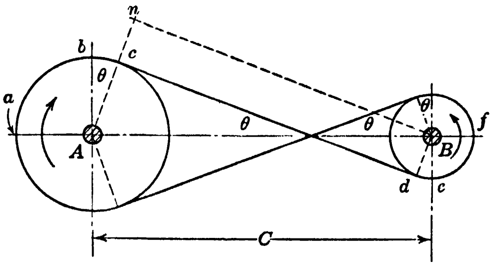


FIG. 154

Crossed belts. In Fig. 154, the diameters of the two pulleys, D_a and D_b on A and B, and the distance C between shafts are known.

To derive the exact formula for L, the length of the belt, we write first

$$L = 2(ab + bc + cd + de + ef), \tag{70}$$

$$ab = \frac{\pi D_a}{4}, \qquad bc = \frac{D_a \theta}{2}.$$

Drawing Bn parallel to cd, we have

$$cd = Bn = AB \cos \theta = C \cos \theta, \qquad de = \frac{D_b \theta}{2}, \qquad ef = \frac{\pi D_b}{4}.$$

Hence we may write

$$L = 2\left(\frac{\pi D_a}{4} + \frac{D_a \theta}{2} + C \cos \theta + \frac{D_b \theta}{2} + \frac{\pi D_b}{4}\right), \tag{71}$$

$$= \frac{\pi D_a}{2} + D_a \theta + 2C \cos \theta + D_b \theta + \frac{\pi D_b}{2}, \tag{72}$$

or

$$L = \left(\frac{\pi}{2} + \theta\right)(D_a + D_b) + 2C \cos \theta. \tag{73}$$

Where

$$\sin \theta = \frac{D_a + D_b}{2C}, \qquad \cos \theta = \sqrt{1 - \frac{(D_a + D_b)^2}{4C^2}}.$$

155. Stepped Pulleys. A ***stepped pulley*** (often called a coned pulley) is a casting composed of two or more pulleys of different diameters. Stepped pulleys are arranged in pairs, one on the drive-shaft and the other on the driven shaft, in such a manner that the same belt may be shifted from one pair of steps to another, Fig. 155. In this way it is possible to obtain different speeds on the driven shaft from a constant speed on the drive-shaft. Stepped pulleys can be designed to operate with either an open or a crossed belt.

156. Stepped Pulleys for Open Belts. Stepped pulleys are designed so that a single belt can be shifted from one pair of steps to another; therefore the pairs of pulleys must be designed for the same length of belt.

The length of the belt on D_1 and d_1, Fig. 155, must be equal to the length of the belt on D_2 and d_2. Then

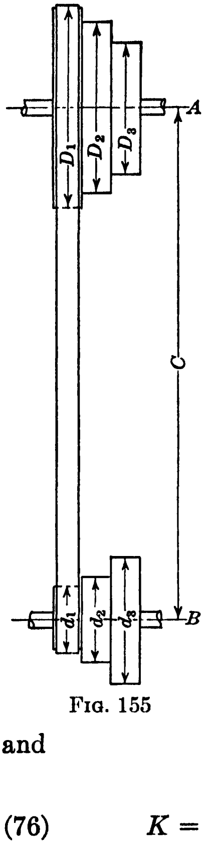


Fig. 155

$$\frac{\pi}{2}(D_1 + d_1) + 2C + \frac{(D_1 - d_1)^2}{4C} = \frac{\pi}{2}(D_2 + d_2) + 2C + \frac{(D_2 - d_2)^2}{4C},$$

or

$$(74)\quad \frac{\pi}{2}(D_1 + d_1) + \frac{(D_1 - d_1)^2}{4C} = \frac{\pi}{2}(D_2 + d_2) + \frac{(D_2 - d_2)^2}{4C}.$$

Since the values of D_1, d_1, and C are known, the left side of equation (74) becomes a constant K. We have also

$$\frac{D_2}{d_2} = \frac{N_{b2}}{N_a}, \quad \text{or} \quad D_2 = d_2\frac{N_{b2}}{N_a};$$

hence, substituting for D_2, we may write the equation in the form

$$(75)\quad K = \frac{\pi}{2}d_2\left(\frac{N_{b2}}{N_a} + 1\right) + \frac{d_2^2\left(\frac{N_{b2}}{N_a} - 1\right)^2}{4C}$$

and

$$(76)\qquad K = \frac{\pi}{2}d_3\left(\frac{N_{b3}}{N_a} + 1\right) + \frac{d_3^2\left(\frac{N_{b3}}{N_a} - 1\right)^2}{4C}.$$

The following example will illustrate the method of calculating the steps.

Example. The shaft A in Fig. 155 rotates at 100 r.p.m. and drives shaft B. Shafts A and B are 64 inches apart. The shaft B is to rotate at 300, 200, and 100 r.p.m. The largest step, D, of the pulley on A is 24 inches in diameter. Find the diameters for the pulley on B and the diameters of the remaining steps on A.

The largest step on the shaft A will give the greatest speed on the shaft B, since the speeds are inversely proportional to the diameter. We have

$$\frac{D_1}{d_1} = \frac{300}{100} = \frac{24}{d_1}, \quad d_1 = \frac{24}{3} = 8 \text{ in.}$$

The value of N_b corresponding to d_2, d_3, etc. must be used in the above equations. Substituting these values, we find

$$K = \frac{\pi}{2}(24 + 8) + \frac{(24 - 8)^2}{4 \times 64} = 51.25,$$

and

$$51.25 = \frac{\pi}{2} d_2 \left(\frac{200}{100} + 1 \right) + \frac{d_2^2 \left(\frac{200}{100} - 1 \right)^2}{4 \times 64},$$

$$51.25 = \frac{3\pi}{2} d_2 + \frac{d_2^2}{4 \times 64},$$

whence

$$d_2 = 10.87 \text{ in.},$$

$$D_2 = d_2 \frac{N_b}{N_a} = 10.87 \times \frac{200}{100} = 21.74 \text{ in.}$$

In like manner

$$51.25 = \frac{\pi}{2} d_3 \left(\frac{100}{100} + 1 \right) + \frac{d_3^2 \left(\frac{100}{100} - 1 \right)^2}{4 \times 64},$$

whence

$$51.25 = \pi d_3, \quad d_3 = 16.30 \text{ in.}, \quad \text{and} \quad D_3 = 16.30 \text{ in.}$$

If $C \geqq 10(D-d)$, the step diameters may be found with negligible error by the method given in the next article for crossed belts.

157. Stepped Pulleys for Crossed Belts. The same belt is shifted from one pair of steps to another on stepped pulleys designed for crossed belts in the same manner as with open belts; the calculations, however, can be simplified.

Since the length of belt for pulleys D_1 and d_1 and pulleys D_2 and d_2 are equal, we have

$$(77) \quad \left(\frac{\pi}{2} + \theta_1 \right) (D_1 + d_1) + 2C \cos \theta_1 = \left(\frac{\pi}{2} + \theta_2 \right) (D_2 + d_2) + 2C \cos \theta_2.$$

Since the angle θ depends on D, d, and C, the equations are identical if $D_1 + d_1 = D_2 + d_2$. Therefore with *stepped pulleys for crossed belts, the sum of the diameters of corresponding steps remains constant.*

The same data used for open belts will be used for crossed belts. We may then write

$$\frac{24}{d_1} = \frac{300}{100}, \quad d_1 = 8 \text{ in.},$$

$$D_1 + d_1 = 24 + 8 = 32 \text{ in.},$$

$$\frac{D_2}{d_2} = \frac{200}{100}, \quad D_2 = 2d_2,$$

$$D_2 + d_2 = D_1 + d_1 = 32 \text{ in.},$$

$$3d_2 = 32, \quad \text{or} \quad d_2 = 10.67 \text{ in.}, \quad \text{and} \quad D_2 = 21.34 \text{ in.},$$

$$\frac{D_3}{d_3} = \frac{100}{100}, \quad D_3 = d_3, \quad D_3 + d_3 = 32, \quad d_3 = 16 \text{ in.}$$

Note that for *crossed belts the step-pulley diameters do not change with the center distance C, while the diameters do change with changes in C when open belts are used.*

Stepped pulleys are frequently designed so that the speeds of the driven shaft increase in a geometric ratio as the belt is shifted from the first step to the succeeding steps. Then with the speed N_a of the drive-shaft constant, the speeds N_{b1}, N_{b2}, etc. of the driven shaft may be found from the equations

$$(78) \qquad N_{b1} = N_a \frac{D_{a1}}{D_{b1}}, \qquad N_{b2} = KN_{b1},$$

$$N_{b3} = K^2N_{b1}, \qquad N_{b4} = K^3N_{b1}, \cdots \text{etc.}$$

158. Equal Stepped Pulleys. It is usually desirable to make the stepped pulleys for the driving and driven shafts the same size. In this way, the pulleys are cheaper to construct, since they can be cast from the same pattern, and erection is simplified as well. In the design of equal-step pulleys, it is necessary to limit the speeds of the driven shaft to the following relation:

The speed of the drive shaft is a mean proportional between the speeds corresponding to the steps on the driven shaft equally spaced on either side of the center step, which may be proved as follows.

If the two step pulleys in Fig. 155 are made equal, D_1 will be equal to d_3, D_2 equal to d_2, and D_3 equal to d_1. With a constant angular velocity for the driving shaft A, the angular velocities for the driven shaft B will be

$$(79) \qquad N_{d1} = \frac{D_1}{d_1} N_a, \quad N_{d2} = \frac{D_2}{d_2} N_a, \quad N_{d3} = \frac{D_3}{d_3} N_a.$$

Hence

$$(80) \qquad \frac{N_{d1}}{N_{d2}} = \frac{D_1 d_2}{d_1 D_2}, \quad \text{and} \quad \frac{N_{d2}}{N_{d3}} = \frac{D_2 d_3}{d_2 D_3}.$$

Since $D_1 = d_3$, $D_2 = d_2$, and $D_3 = d_1$, we may write

$$(81) \qquad \frac{N_{d1}}{N_{d2}} = \frac{N_{d2}}{N_{d3}}, \quad \text{or} \quad \frac{N_{d1}}{N_a} = \frac{N_a}{N_{d3}}.$$

A crossed belt or an open belt may be used with equal stepped pulleys, and the diameters of the steps can be calculated by the methods explained in §§ 156 and 157.

159. Speed Cones. It is sometimes desirable to use speed cones instead of stepped pulleys to transmit motion between two shafts. The cones may be designed for crossed belts or for open belts. If a crossed belt is used, the pulleys become frustra of cones; if open belts are used, the pulleys become conoids. The following examples will illustrate these two cases.

160. Speed Cones for Crossed Belts. In Fig. 156, the shaft A is the driver and rotates at 100 r.p.m. The smallest diameter of the cone is 10 inches. Determine the size of equal speed cones that will vary the speed of the driven shaft from 50 r.p.m. to 200 r.p.m.

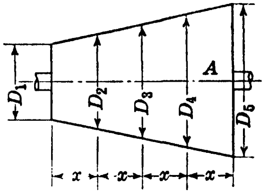


FIG. 156

If the driver rotates at 100 r.p.m., and the driven shaft varies between 50 and 200 r.p.m., one end of the cone must be twice as large as the other. Since the cones are designed for a crossed belt, the sum of the diameters for any position of the belt must be equal. That is to say, the same conditions obtain for the design of speed cones as for stepped pulleys.

Lay off five lines, or any other convenient number, D_1, D_2, D_3, D_4, and D_5 perpendicular to the shaft A, Fig. 156, and any desired distance, x, apart. D_1 is 10 inches, and D_5 is 20 inches. The values for the diameters at D_2, D_3, and D_4 form an arithmetical progression between 10 and 20, and the line drawn through the ends of D_1, D_2, etc. is, therefore, a straight line. The following table gives the values for the different diameters and the corresponding speeds for the driven shaft. Figure 156 gives the construction for the cone on shaft A. The cone on shaft B will be the same as the cone on A.

R.P.M.	Driver	Driven	R.P.M.
100	$D_1 = 20$	$d_1 = 10$	200
100	$D_2 = 17.5$	$d_2 = 12.5$	140
100	$D_3 = 15$	$d_3 = 15$	100
100	$D_4 = 12.5$	$d_4 = 17.5$	71.43
100	$D_5 = 10$	$d_5 = 20$	50

It will be noted that the requirements for equal stepped pulleys have been met in the above design.

161. Speed Cones for Open Belts. GIVEN. The same data as given above for crossed belts. The shafts in Fig. 157 are 50 inches apart.

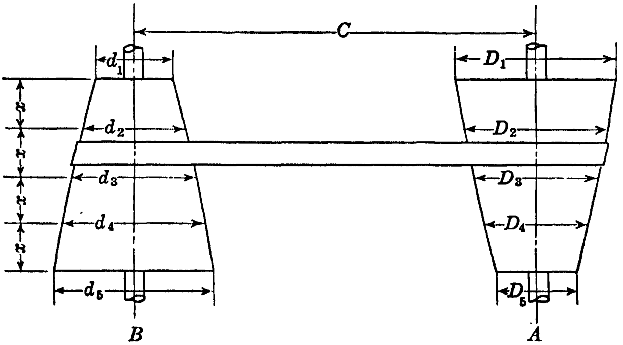


FIG. 157

REQUIRED. Find the same quantities as for crossed belts.

CONSTRUCTION. Lay out two cones for a crossed belt, using five points as in the previous examples, and calculate the five speeds of the driven shaft, that is, when the belt is shifted from

D_1 to D_2, etc. As the belt is shifted, the speed ratios will remain the same for an open belt as for the crossed belt of the preceding problem. Therefore, with the speed ratio known for five points, and the diameter of one end of the conoid known, the other diameters can be calculated by the same method used for equal stepped pulleys for open belts. The following table gives the values for the different diameters and the corresponding speeds.

R.P.M.	Driver	Driven	R.P.M.
100	$D_1 = 20$	$d_1 = 10$	200
100	$D_2 = 18.02$	$d_2 = 12.87$	140
100	$D_3 = 15.16$	$d_3 = 15.16$	100
100	$D_4 = 12.87$	$d_4 = 18.02$	71.43
100	$D_5 = 10$	$d_5 = 20$	50

No allowance for belt width has been made in the above problems; therefore it will be necessary to extend the cone past D_1 and D_5 so that the center-line of the belt will be at D_1 or D_5, or else to redesign the cone for a given belt width.

It will be necessary in any speed cone mechanism to have a shifter that will hold the belt in place on the cones; otherwise the belt will tend to climb towards the high side of the cones.

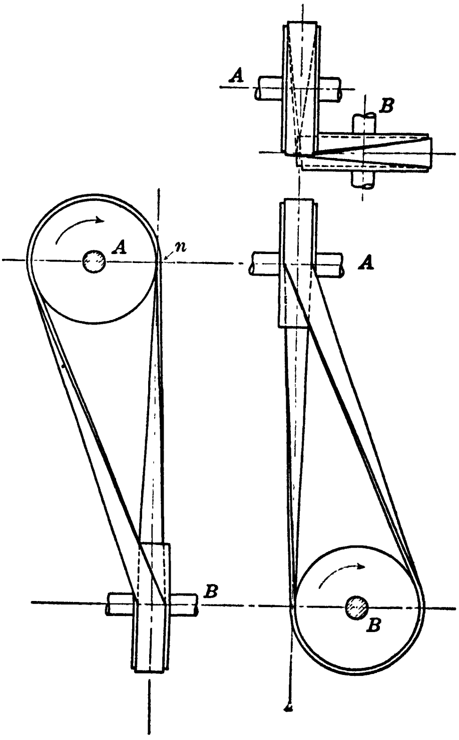
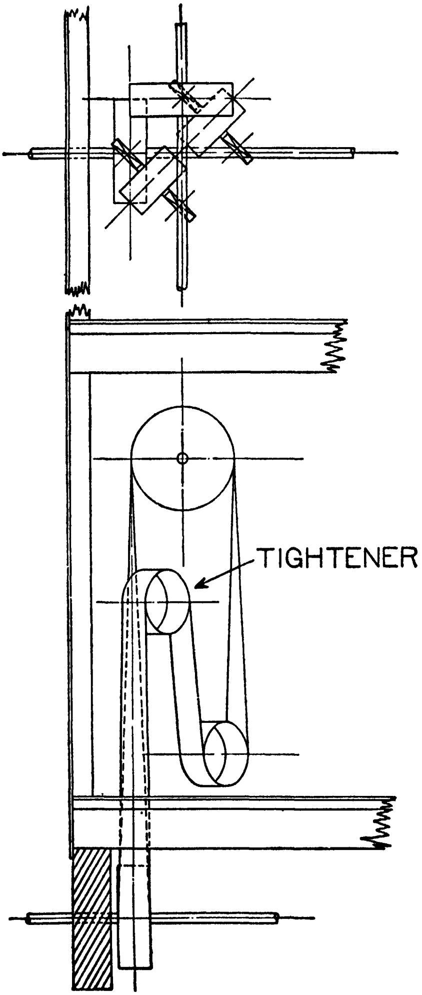


Fig. 158

162. Belts for Shafts Not in the Same Plane. When it is necessary to connect two shafts which are neither parallel nor intersecting, the following fundamental law governing the location of the pulleys on the shaft must be closely adhered to. *The line at which the neutral surface of a belt leaves a pulley must lie in a plane which bisects the face of the pulley onto which the belt runs.* A ***quarter-turn belt*** is a good illustration of this type of belting. Figure 158 shows three views of two pulleys connecting the shafts *A* and *B* by a quarter-turn belt. Note that a plane passed through *n*, the point where the neutral surface leaves the pulley on *A*, bisects the face of the pulley on *B* on which the belt is running. This drive is not reversible.

The quarter-turn belt drive may be made reversible by the use of one or more idler pulleys. Figure 159 indicates a reversible drive with two idler pulleys, one of which is used as a tightener.



American Tool and Machine Co., Boston, Mass.

FIG. 159. REVERSIBLE QUARTER TURN BELT WITH TIGHTENER.

163. Intersecting Shafts. When belting is used to connect intersecting shafts, it is necessary to make use of additional pulleys known as ***idler pulleys,*** or ***mule pulleys.*** An idler pulley is a pulley inserted between two other pulleys to guide the belt properly from one to the other. An idler pulley is also used to take up the slack in a belt between two pulleys, in which case it is called a ***tension pulley*** or ***belt tightener.***

Figure 160 shows three views of the pulleys and belt used to connect two shafts A and B, which intersect at 90°. The pulleys C and D are idler pulleys. This drive is reversible.

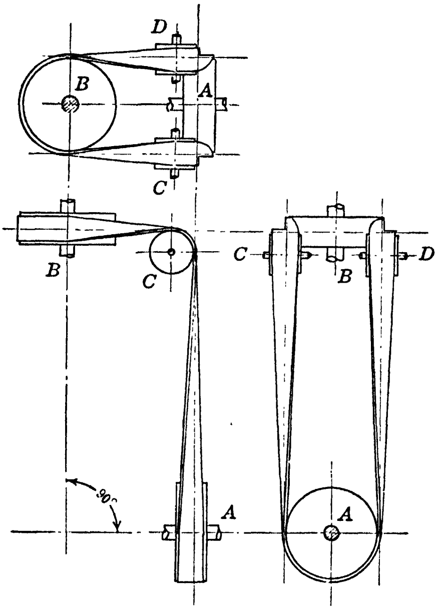
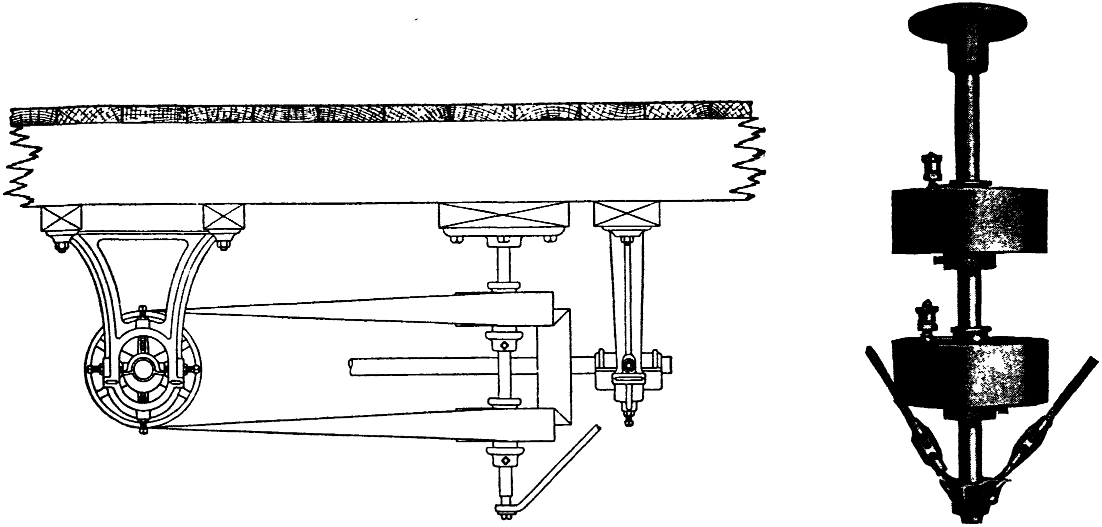


Fig. 160

Either a stationary or adjustable ***mule stand*** may be substituted for the idler pulleys C and D in Fig. 160. Figure 161 shows the stationary mule stand and its application, and Fig. 162 shows the adjustable mule stand and its application.

164. Crown Pulleys. A pulley is said to have a ***crown*** when the center of the face of the pulley is higher, or has a greater radius, than the two edges. The line drawn from the center of the pulley face to the edges may be straight or curved. A crown is used on a pulley to allow for any small misalignment that might occur in placing the pulleys on the shafts. The high point in



American Tool and Machine Co., Boston, Mass.

(a)

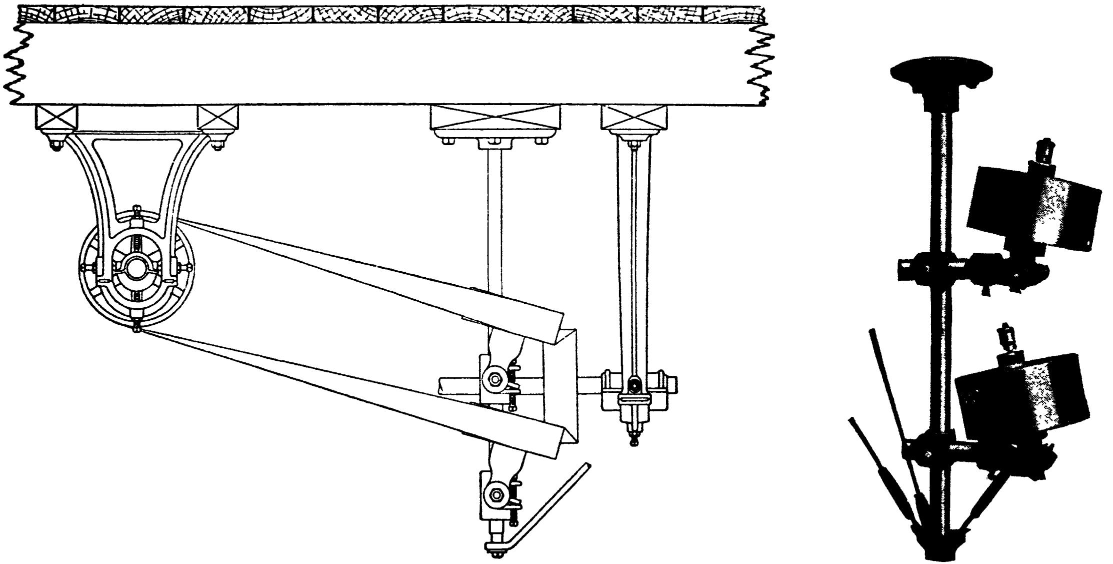
QUARTER TURN BELT AND MULE PULLEYS.

Dodge Mfg. Co., Mishawaka, Ind.

(b)

MULE STAND.

FIG. 161



American Tool and Machine Co., Boston, Mass.

(a)

QUARTER TURN BELT AND ADJUSTABLE MULE PULLEYS.

Dodge Mfg. Co., Mishawaka, Ind.

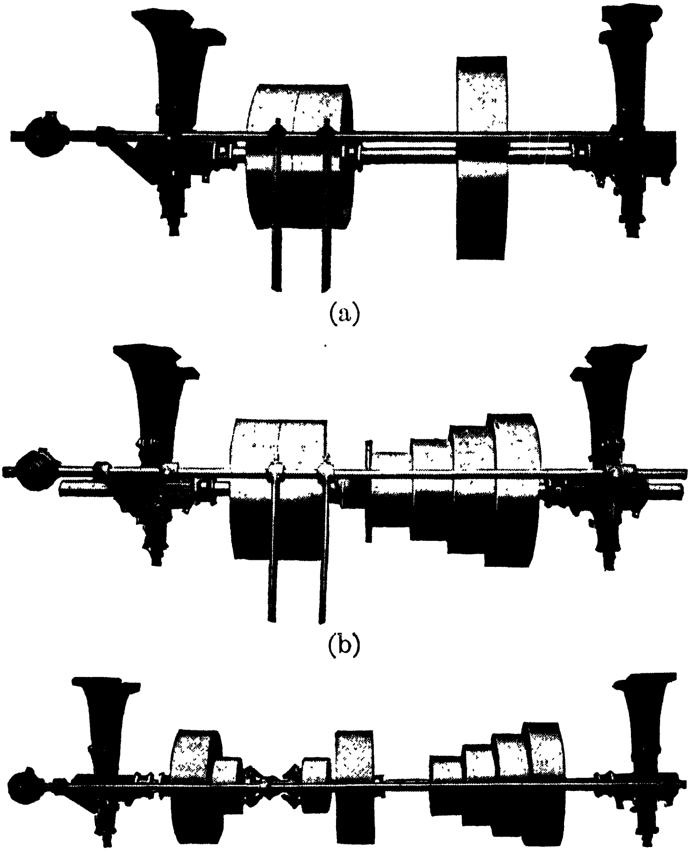
(b)

ADJUSTABLE MULE STAND.

FIG. 162

the center of the face of the pulley tends to cause the belt to center on the pulley.

165. *Tight* and *Loose* Pulleys. A ***tight pulley*** is one that is fixed or keyed to a shaft, and a ***loose pulley*** is one that is free to turn on a shaft. A loose pulley is used side by side with a tight pulley so that the belt can be shifted from one to the other and can throw the driven shaft in or out of motion without interfering with the speed of the drive shaft.



(a)

(b)

Dodge Mfg. Co., Mishawaka, Ind.

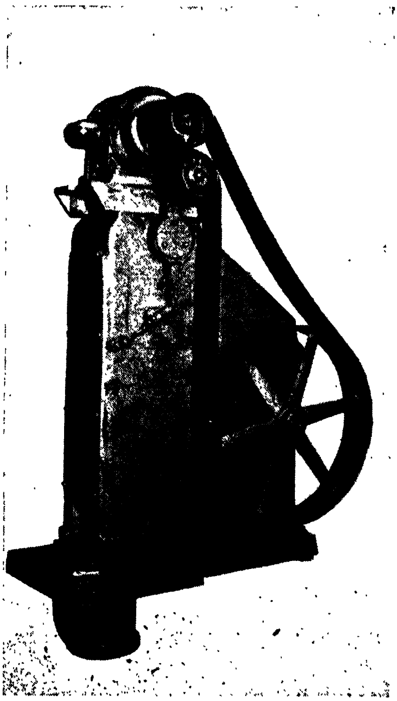
(c)

Fig. 163

Figure 163a shows the ***countershaft*** arrangement for a single drive and Fig. 163b for a step-pulley drive using tight and loose pulleys.

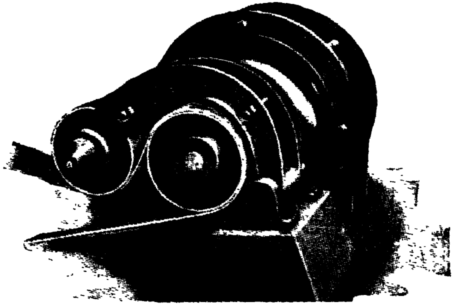
Two pulleys, one for the forward drive and one for the reverse drive, with a friction clutch as illustrated in Fig. 163(c) may be substituted for tight and loose pulleys.

166. Short Center Drives. Where a belt is used for a power drive between a small pulley and a large pulley with a short center distance, the power transmitted depends on the arc of contact of the belt on the small pulley. To increase this arc and thereby increase the power transmitted, an idler or guide pulley is used on the slack or low tension side of the belt. This idler should be located close to the small pulley and may be held



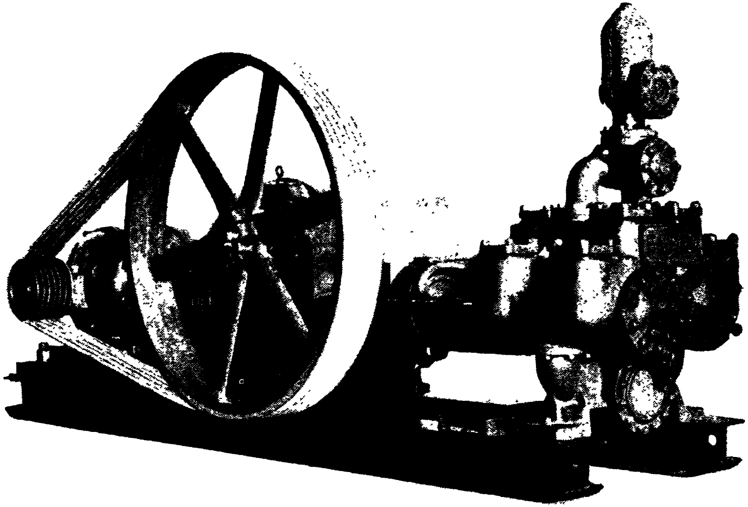
American Well Works, Aurora, Ill.

FIG. 164. SHORT-CENTER BELT DRIVE, SPRING LOADED TENSION PULLEY.



Stephens-Adamson Mfg. Co., Aurora, Ill.

FIG. 165. SHORT CENTER BELT DRIVE, WEIGHTED TENSION PULLEY.



Gates Rubber Co., Denver, Colo.

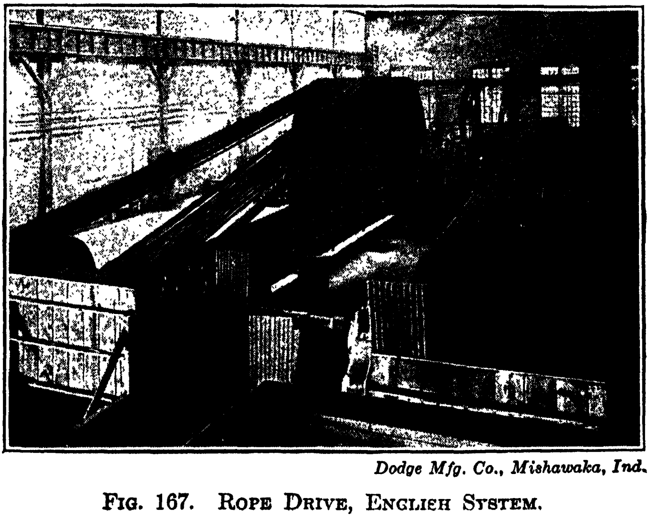
FIG. 166. SHORT CENTER DRIVE WTIH V-BELT.

against the belt by springs or by weights. Figures 164 and 165 show common arrangements for this type of drive.

The V-belt as illustrated in Fig. 166 is becoming very popular for short center drives since it eliminates the small idler pulley.

167. Ropes. Ropes used for the transmission of power are generally made of hemp or cotton, or of steel wire, and any of these may be substituted for belting. Hemp rope will give satisfactory service when transmitting power between shafts as much as 175 feet apart without the use of intermediate idlers, and wire rope can be used to connect shafts with a center distance of 1800 feet. Wire rope, however, is generally used for hoisting, and its use in long-distance-power transmission is almost a thing of the past because of the application of electricity for this purpose.

Rope, made of hemp or similar materials, is used for the transmission of power, and there are two general systems of rope drives in use, often called, respectively, the ***American system*** and the ***English system.*** If a continuous rope is wound around the driver and driven pulley, the system is known as the American; and if a number of different ropes are used in parallel between the driver and driven pulley, the system is known as the English. The English system is usually considered better for heavy main



Dodge Mfg. Co., Mishawaka, Ind.

FIG. 167. ROPE DRIVE, ENGLISH SYSTEM.

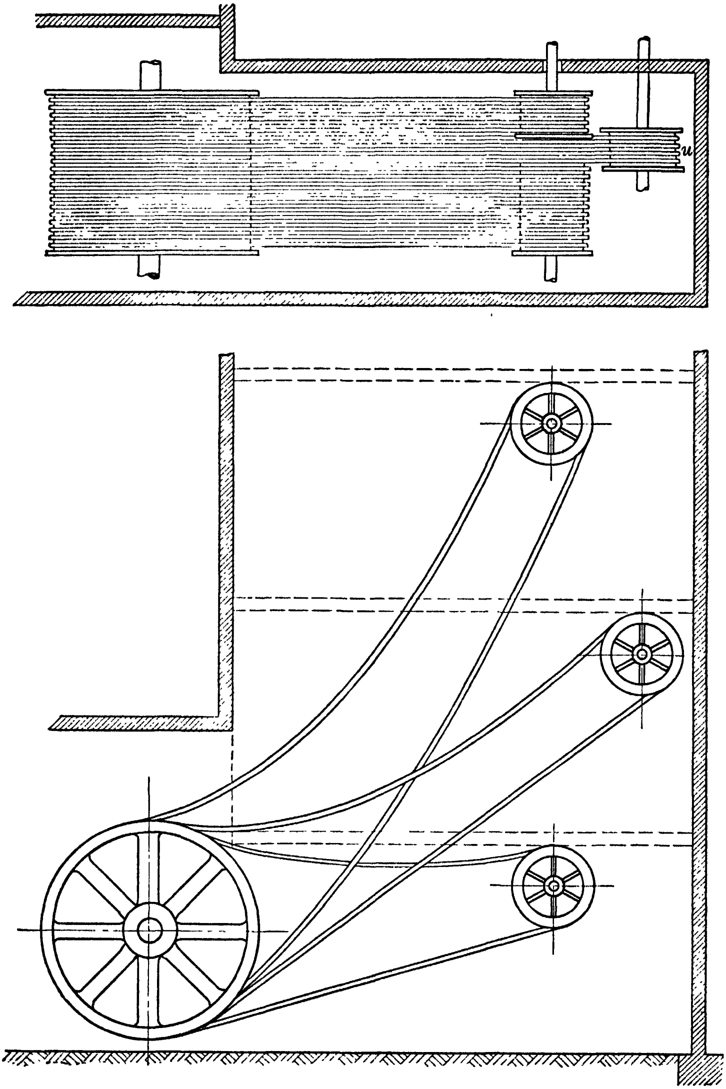
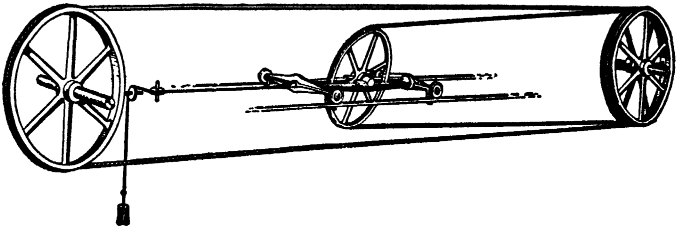


FIG. 168. ENGLISH SYSTEM ROPE-DRIVE FOR A THREE-STORY FACTORY.

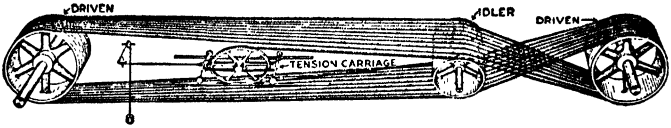
drives, and the American system better for the auxiliary drives, but there are advantages and disadvantages for each drive, and the student will do well to consult technical papers and texts on Machine Design on this subject.



Stephens-Adamson Mfg. Co., Aurora, Ill.

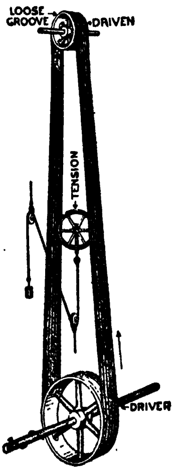
Fig. 169. Plain Horizontal Drive, American System.

The English system is illustrated in Figs. 167 and 168, and the American system is illustrated in Figs. 169 and 170. Note the guide sheaves used in the American system to guide the rope from the last groove on the driven pulley to the first groove on the driver. Note also the weights to keep the rope tight on the pulleys.



Stephens-Adamson Mfg. Co., Aurora, Ill.

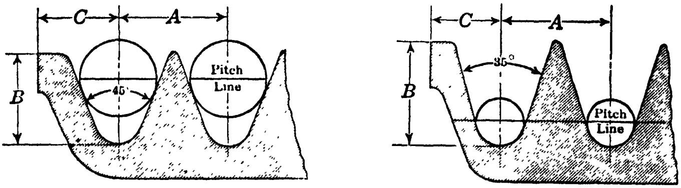
Horizontal Reverse Drive.



Vertical Drive, American System.

Fig. 170

The best speed at which to run a manila rope drive is considered to be around 4000 to 4500 ft. per minute. Cotton ropes may be run at considerably higher speeds.



(a)
AMERICAN SYSTEM.

(b)
ENGLISH SYSTEM.

FIG. 171. GROOVES FOR ROPE SHEAVES.

A rope uses sheaves with one or more grooves to transmit power between shafts, and the shape of the groove cut in the sheave depends on the system as well as on the type of rope used. Figure 171 illustrates the shape of groove cut in the sheave for the American system and for the English system when a manila rope is used. Figure 172 illustrates a sheave for a wire rope.

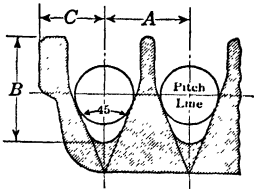


FIG. 172. GROOVES FOR MARLIN-COVERED WIRE ROPE.

168. Rope Blocks. A rope block, also known as a block-and-tackle or pulley block, is a mechanism consisting of two parallel shafts, each shaft having one or more pulleys which are free to turn with the shaft. A rope is fastened to one of the shafts and wound around the pulleys, alternating between the shafts, leaving a free end, Fig. 173. If one of the shafts is hung from a support, the other shaft can be moved by taking up or letting out the free end of the rope.

In Fig. 174, the shaft *a–b* is hung from a support, and the bar *d–e* is supporting the weight *W*. The pulleys *A* and *B* are fixed to the bar a–b, and the pulley c is fixed to the bar d–e. A rope is fastened to the bar d-e at d, and wound around the pulleys as shown, leaving the free end F. Suppose a pull at F

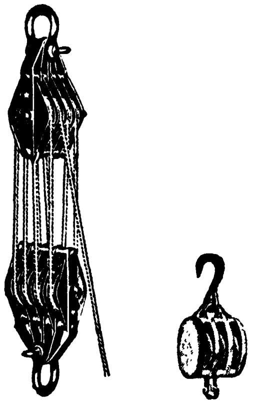

(a) (b)

FIG. 173

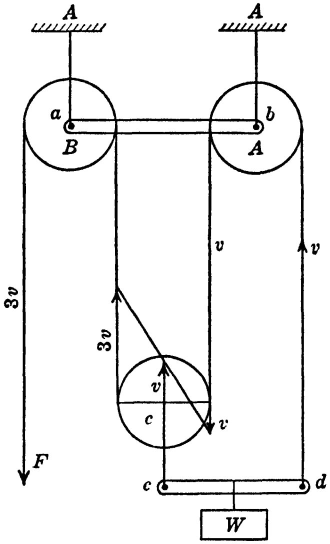


FIG. 174

causes the bar d–e to raise with a velocity v. The rope at d will have a linear speed of v, and as it comes over the pulley a, it will still have a linear speed v. As the rope goes on the pulley c, it has a linear speed of v down, and the pulley c has a velocity v up; therefore the rope leaves the pulley c with a linear speed $3v$ up (construction shown in the figure). The linear speed of the rope is still $3v$ after passing over the pulley b. If a second pulley is added at the bottom, the rope will have a speed of $3v + 2v$ or $5v$; and if a third pulley is added, the rope will have a speed of $7v$, and so on. The speed of the last strand of rope, therefore, may be found from the equation,

$$v_F = (2n + 1)v_W, \tag{82}$$

where n denotes the number of moving pulleys. If the fixed end of the rope is on the stationary block, the equation becomes

$$v_F = 2nv_W. \tag{83}$$

In the system shown in Fig. 174, we have

$$v_F = (2 \times 1 + 1)v_W = 3v_W.$$

Therefore

$$\frac{\text{linear speed } F}{\text{linear speed } W} = \frac{3}{1}.$$

Disregarding friction, we may write

$$Wv_w = Fv_f, \quad Wv = F \times 3v, \quad \frac{W}{F} = \frac{3}{1},$$

where W = weight, lifted, v_w = velocity of weight, F = force applied, v_F = velocity of force.

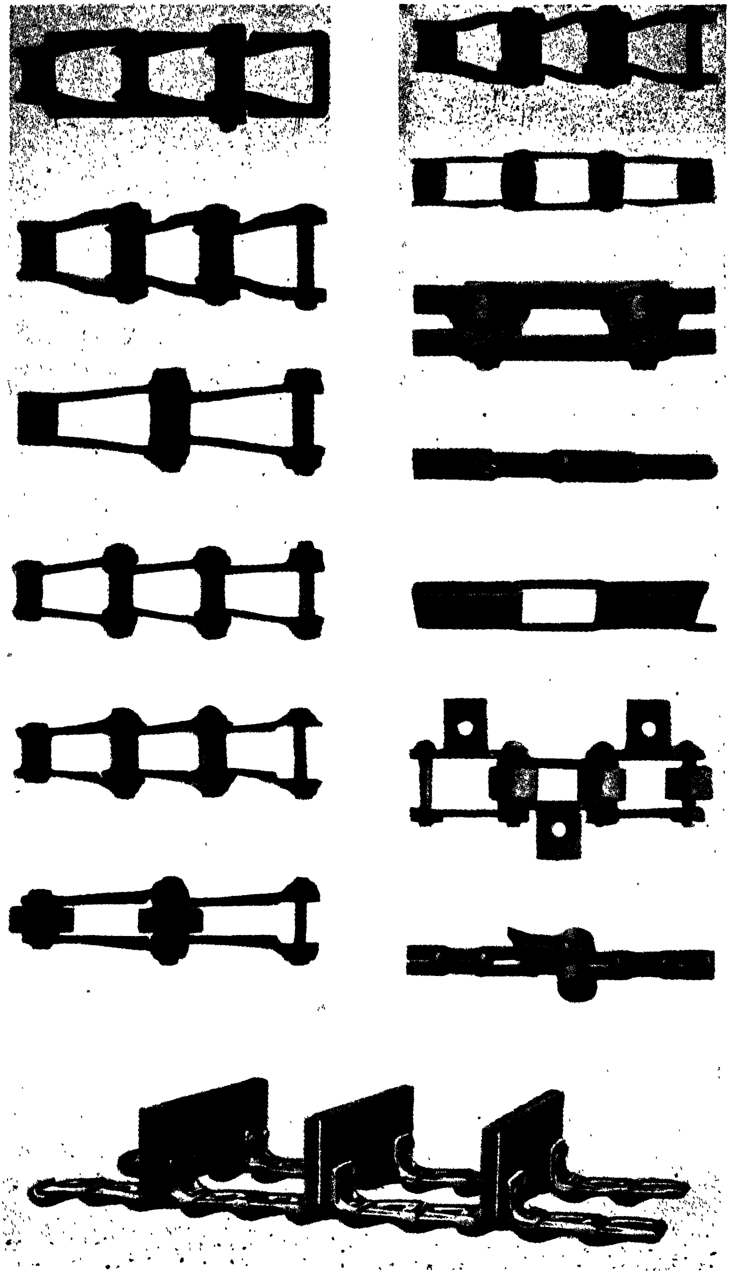
The ***mechanical advantage*** of a rope block is *equal to the weight lifted divided by the force applied, disregarding friction,* or to *the velocity of the force divided by the velocity of the weight.*

The problem can be solved by raising the weight W, Fig. 174, one foot, which will give a foot of slack on each side of the pulley c and at d. Therefore it will be necessary to pull over 3 feet of rope at F in order to take up the slack. The ratio of the linear displacement of F to W is 3, which is also the mechanical advantage.

169. Chains. Chains are used for transmitting power, for conveying and elevating various products, and for hoisting purposes. The type and design of the chain depend upon the purpose for which it is to be used.

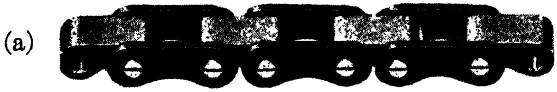
Chains for power transmission are used where it is necessary to have positive action between the driver and driven shafts, or where the distance between shafts is too short for successful belt drives. Conveyor chains, block chains, roller chains, and silent chains are used to transmit power; and the selection of the proper drive will depend upon the design and conditions of use.

170. Chain Drives. ***Conveyor chains,*** Fig. 175, are cast or roughly machined, and are not suitable for power transmission when the chain speed exceeds 600 feet per minute. ***Block chains,*** Fig. 176a, may be used for chain speeds up to 900 feet per minute. ***Roller chains,*** Fig. 176b, may be used for chain speeds up to 1200 ft. per minute. ***Silent chains,*** Figs. 177 and 178, may be used

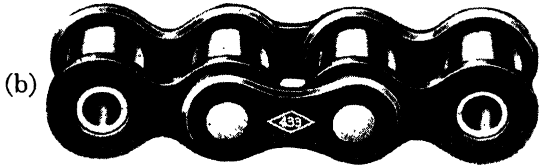


Stephens-Adamson Mfg. Co., Aurora, Ill.

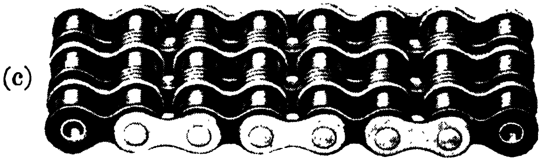
FIG. 175. CONVEYOR CHAINS.



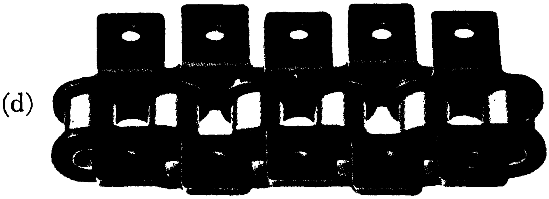
BLOCK CHAIN.



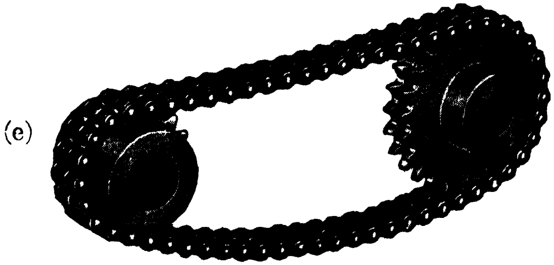
SINGLE ROLLER CHAIN.



TRIPLE ROLLER CHAIN.



SPECIAL PURPOSE ROLLER CHAIN.

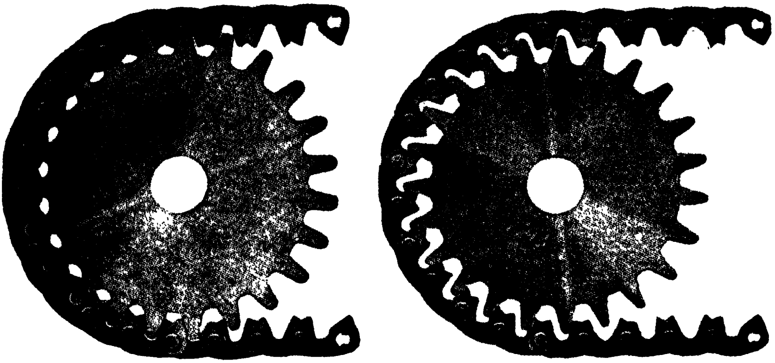


Diamond Chain and Mfg. Co., Indianapolis, Ind.

DOUBLE ROLLER CHAIN AND SPROCKETS.

FIG. 176

for chain speeds up to 1500 feet per minute. Higher speeds may be used, especially with small sprocket sizes, but an increase in speed will decrease the useful life of the chain.



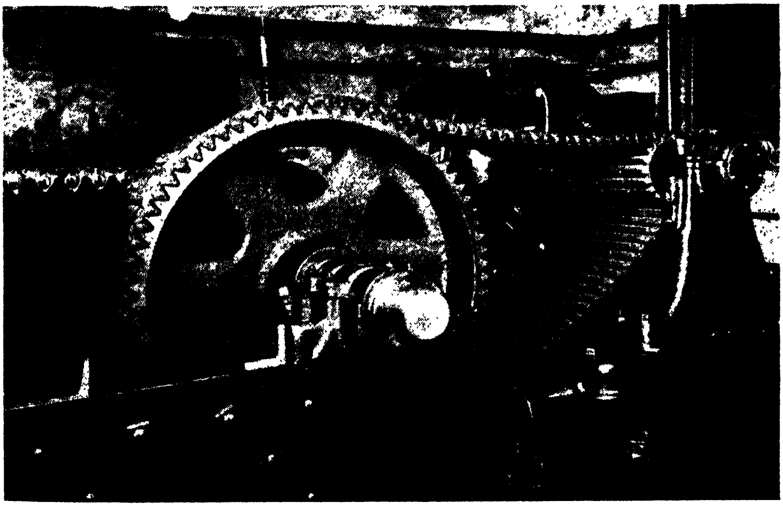
Link-Belt Co., Indianapolis, Ind.

(a) SILENT CHAIN ON NEW SPROCKET.

(b) SILENT CHAIN ON OLD SPROCKET.

FIG. 177

The ***angular-speed ratio*** *of chain-connected shafts depends on the number of teeth on the chain sprockets and not on the pitch diameter*



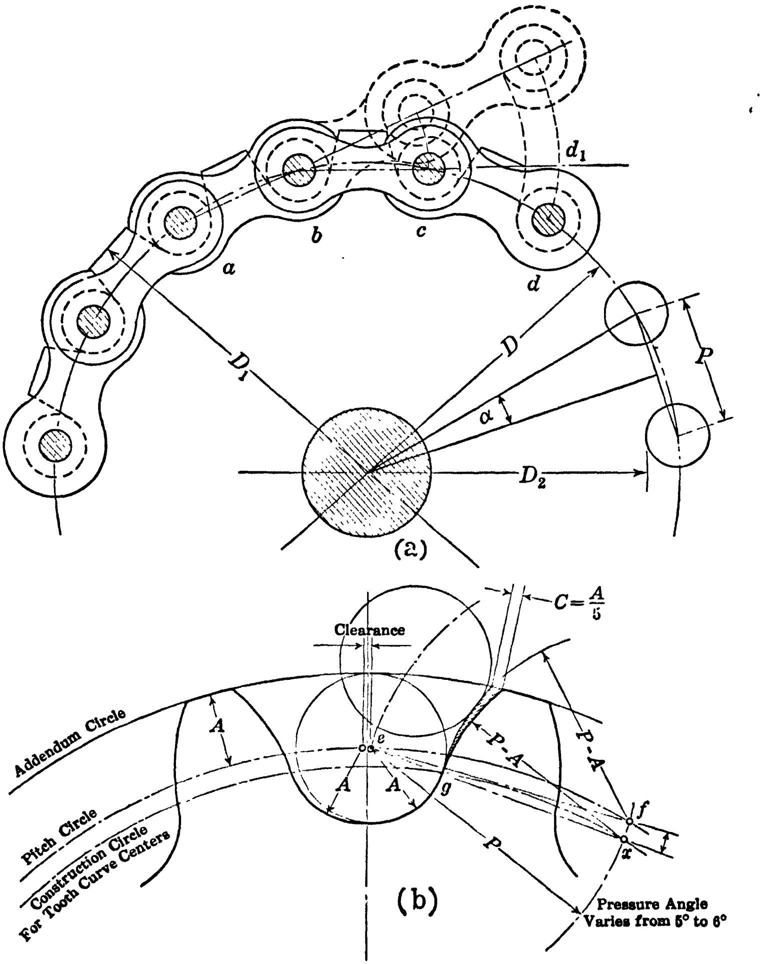
Morse Chain Co., Ithaca, N. Y.

FIG. 178. SILENT CHAIN DRIVE.

of the sprockets. We have

$$\frac{N_a}{N_b} = \frac{T_b}{T_a}, \tag{84}$$

where N_a = r.p.m. of drive sprocket, N_b = r.p.m. of driven sprocket, T_a = number of teeth in driving sprocket, T_b = number of teeth in driven sprocket.



Diamond Chain and Mfg. Co., Indianapolis, Ind.

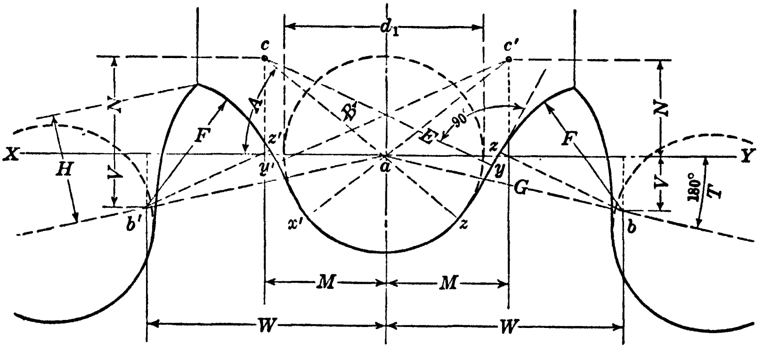
FIG. 179. OLD FORM OF SPROCKET TOOTH.

The shape of the teeth on a given sprocket wheel will depend upon the type of chain used. Figure 179a shows the construction of the tooth curves for a roller chain or any other chain having cylindrical surfaces in contact with the teeth.

The ***pitch diameter*** of the sprocket is the diameter of the circle passing through the centers of the rolls or pins of the chain joints. The pitch P of the chain is the distance from center to center of the rolls, Fig. 179a. An inspection of the figure shows that the pitch diameter is

$$D = \frac{P}{\sin \alpha}, \quad \text{and} \quad \alpha = \frac{360^\circ}{2T}, \tag{85}$$

where D = pitch diameter in inches, P = chain pitch in inches, T = number of teeth on sprocket.



Diamond Chain and Mfg. Co., Indianapolis, Ind.

P = Pitch; d = Nominal Roller Dia.; T = Number of Teeth.

$d_1 = 1.005d + .003''$; $A = 35^\circ + \frac{60}{T}$; $B = 18^\circ - \frac{56}{T}$.

$ac = 0.8d$; $M = 0.8d \cos\left(35^\circ + \frac{60}{T}\right)$; $N = 0.8d \sin\left(35^\circ + \frac{60}{T}\right)$

$G = 1.24d$; $W = 1.24d \cos\frac{180}{T}$; $V = 1.24d \sin\frac{180}{T}$

$F = bz = d\left[0.8 \cos\left(18^\circ - \frac{56}{T}\right) + 1.24 \cos\left(17^\circ - \frac{64}{T}\right) - 1.3025\right] - .0015''$

The pressure angle when chain is new is $xab = 35^\circ - \frac{120}{T}$

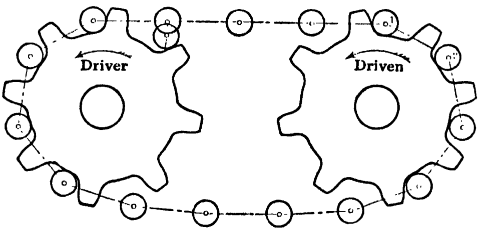
The minimum pressure angle is $xab - B = 17^\circ - \frac{64}{T}$

The average pressure angle is $26^\circ - \frac{92}{T}$

FIG. 180

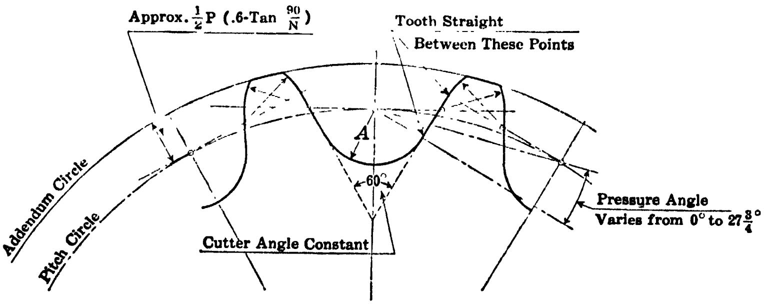
As the link *cd* comes off the sprocket wheel, it rotates about *c* until it cuts a straight line through *bc* at d_1, and then *c* and *d* rotate about *b* until they cut a straight line through *ab*. The other side of the tooth curve is found in the same manner, and the roll is allowed for, as shown. The size of the roll determines the top and bottom of the teeth. Figure 179b shows the way the tooth is rounded off in actual construction to allow for clearance and wear of chain. Figure 180 shows the commercial layout for the old form of sprocket tooth.

Chains will stretch and wear when they are placed in service under load conditions, and the design of the sprocket wheel must be modified to accommodate this increase in length. The sprocket wheels are modified in two different ways. With tooth forms such as the one just developed, the sprocket pitch diameter is slightly increased on the driver and decreased on the driven wheel. The effect of this change is to allow contact between only one roller and tooth at one time, Fig. 181.



Diamond Chain and Mfg. Co., Indianapolis, Ind.

FIG. 181



Diamond Chain and Mfg. Co., Indianapolis, Ind.

FIG. 182. REYNOLD'S TOOTH FORM.

This change places all the load on a single tooth, and to prevent this the Renold's form of tooth, Fig. 182, is often used. These teeth are formed with straight sides, making an angle with the radial lines; and all the rollers around the sprocket are in contact with the teeth at the same time. An increase in chain pitch due to wear or stretching under load merely causes the rollers to move out on the teeth, increasing the pitch diameter.

The teeth on the sprocket wheel used with silent chains are also straight-sided, Fig. 178, and the action is similar to that of the Renold's tooth just described.

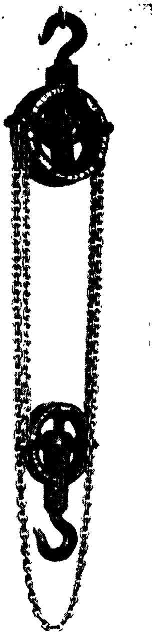
The chain link leaving the driven sprocket is pivoting about one of the pin centers, and the distance of this pin center from the line of centers of the sprockets is changing. This is also true of the chain link entering the driving sprocket. This causes an oscillation of the center-line of the chain, which may be very pronounced with long pitch chains running on sprockets having a small number of teeth. This action will cause a variation in the angular velocity of the driven sprocket during each revolution even if the driver has a constant angular velocity, and for this reason sprockets with small numbers of teeth should be avoided.

171. Conveying and Elevating Chains. Chains for conveying and elevating purposes are of more or less special design, and they appear in a large variety of forms and combinations to meet the needs of industry. The student will do well to consult the manufacturers' catalogs to get some idea of the variety of chains for this purpose. Figure 175 merely indicates some of the possibilities in this field.

172. Hoisting Chains. One of the simplest chain hoists is the ***differential pulley block,*** Fig. 183. The sheaves A and B are fixed together, Fig. 184, and the sheave A is larger than the sheave B. An endless chain is wound around the sheave A, then around C, and up around B, as shown.

If the sheave A makes one revolution, the chain at X will move up a distance of πD_a, and the sheave C will move up a distance of $\pi D_a/2$. But the sheave B is rotating with A; therefore the chain Y will move down a distance of πD_b, lowering the sheave C

a distance of $\pi D_b/2$. The sheave C has moved a distance of $\pi D_a/2 - \pi D_b/2$, while the chain at F has moved a distance of πD_a. Therefore we may write



Yale and Towne Mfg. Co., Phila., Pa.

FIG. 183. DIFFERENTIAL CHAIN BLOCK.

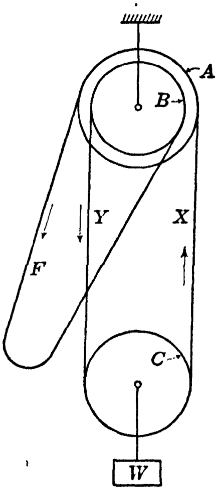


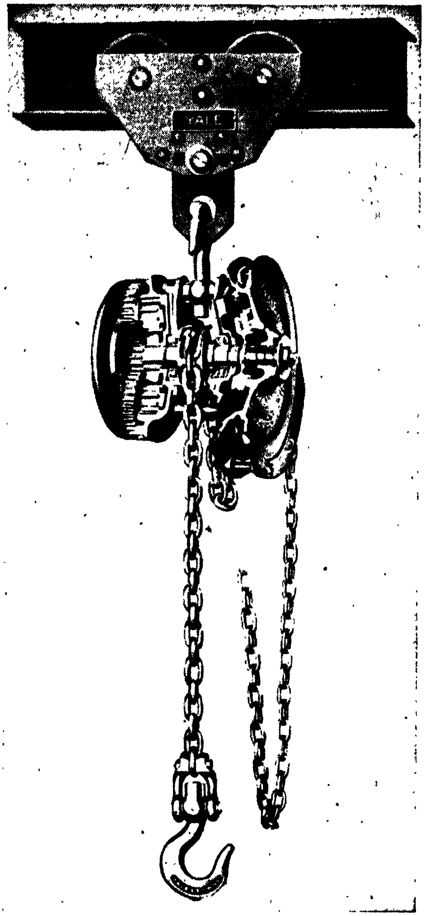
FIG. 184

$$(86)\quad \frac{\text{linear speed } F}{\text{linear speed } W} = \frac{\pi D_a}{\pi(D_a - D_b)/2}$$

$$= \frac{2D_a}{D_a - D_b} = \frac{W}{F} = \text{mechanical advantage,}$$

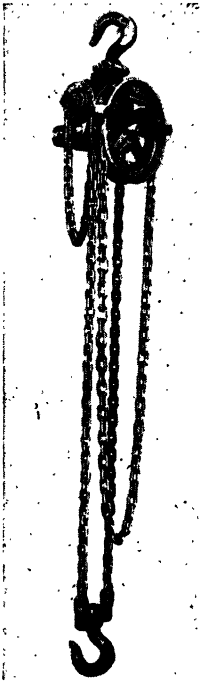
where D_a = diameter of sheave A, D_b = diameter of sheave B. Figures 185 and 186 illustrate two other types of chain hoists that are in general use.

173. Variable Speed Devices. Belts with step pulleys and with cones have been discussed as devices for varying the speed of the driven shaft when the driving shaft remains constant. With step pulleys, the number of driven shaft speeds is limited to the number of steps used on the pulleys. The belt cones or speed cones, used with short center distances, allow any speed variation within the design limits; but because of the twisting of the belt, all but very narrow belts give rise to difficulties, unless some special design feature is made use of such as that shown in Fig. 187.



Yale and Towne Mfg. Co., Phila., Pa.

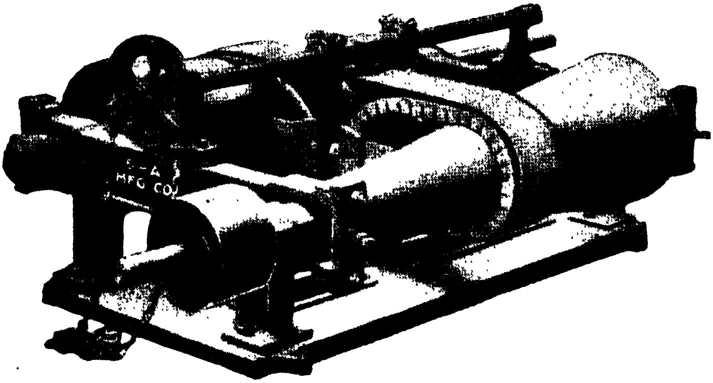
FIG. 185. SPUR GEARED BLOCK.



Yale and Towne Mfg. Co., Phila., Pa.

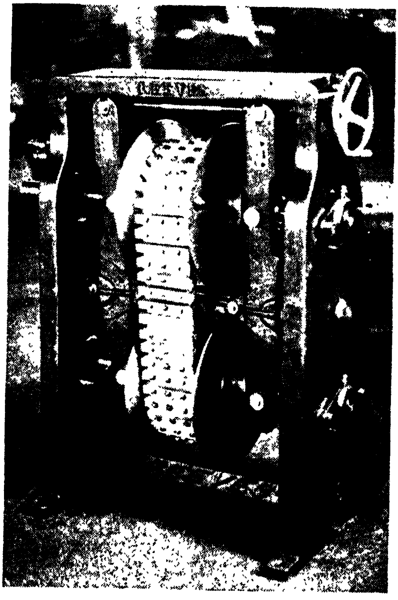
FIG. 186. SCREW GEARED BLOCK.

A very successful variable speed drive is illustrated in Fig. 188. This consists of two pairs of conical discs or cones mounted on parallel shafts and connected by a mechanism so constructed that the cones on one shaft are being separated while the cones



Stephens-Adamson Mfg. Co., Aurora, Ill.

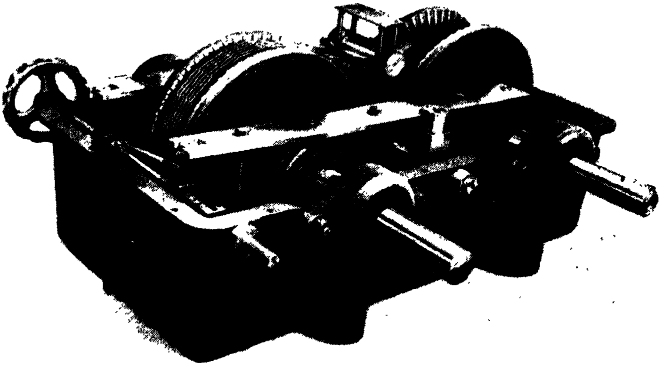
FIG. 187. M. AND W. SPEED CHANGER.



Reeves Pulley Co., Columbus, Ind.

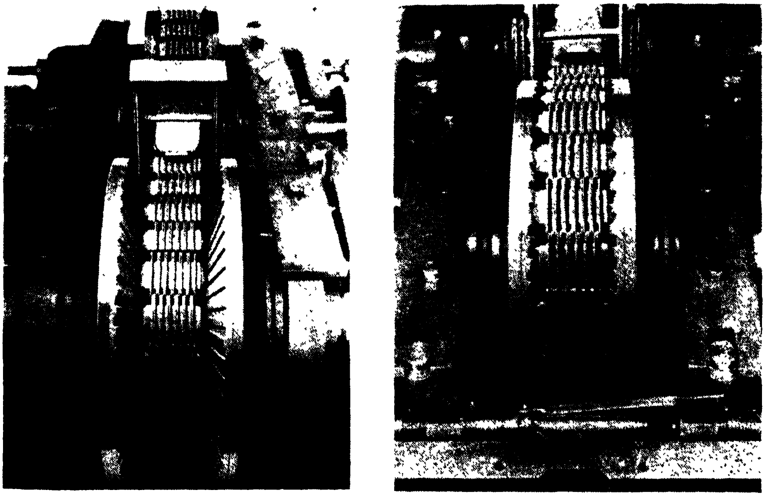
FIG. 188. VARIABLE-SPEED TRANSMISSION.

on the other shaft are being closed. A V-type belt is used to transmit power between the pairs of cones; and any desired speed ratio, within the design limits of the device, may be obtained if the positions of the cones and the belt are varied, the effective diameters of the conical disc pulleys thus being changed. These units have been built with speed ratios varying from 2 to 1 up to 12 to 1.



Link-Belt Co., Indianapolis, Ind.

FIG. 189. LINK-BELT P. I. V. GEAR (Positive Infinitely Variable).



(a) CHAIN ON MINIMUM DIAMETER.

(b) CHAIN ON MAXIMUM DIAMETER.

FIG. 190

A device similar to the one just discussed and known as the P.I.V. variable speed transmission is shown in Figs. 189 and 190. The principal difference between the devices is that a V-type chain has been substituted for the V-belt, and the device is enclosed in an oil-tight case. The faces of the cones are grooved radially, and the chain pins are made of bundles of laminated sheet steel which adjust themselves to fit into the grooves of the cones. In this way, the drive becomes positive instead of depending on the belt friction to transmit power.

PROBLEMS

1. Two shafts are to be connected by a belt and pulley drive. The driving shaft rotates at 250 r.p.m. in a clockwise direction. The driving pulley is 20 in. in diameter.
   - (a) Sketch this drive, and determine the driven pulley size if it is to rotate at 150 r.p.m. in a clockwise direction.
   - (b) Sketch this drive and determine the driven pulley size if it is to rotate at 300 r.p.m. counter-clockwise.
2. Two parallel shafts, 36 in. apart, rotate in the same direction, the driver rotating at 250 r.p.m. and the driven at 600 r.p.m. The smallest pulley is 10 in. in diameter.
   - (a) Sketch a belt drive for this installation, giving the pulley diameters, and indicate which side of the belt is the pulling or high-tension side.
   - (b) What is the belt speed in ft. per min.?
   - (c) What is the arc of contact on each pulley in degrees?
3. Two parallel shafts rotate in opposite directions, the driver rotating at 1800 r.p.m. and the driven at 800 r.p.m. The largest pulley is 36 in. in diameter, and the center distance between shafts is 5 ft.
   - (a) Sketch a belt drive for this installation, giving the pulley diameters, and indicating which side of the belt is the tight or pulling side.
   - (b) What is the belt speed in ft. per min.?
   - (c) What is the arc of contact on each pulley in degrees?
4. A 200 r.p.m. water turbine with a 72-in. pulley drives a generator having an 8-in. pulley on its shaft. The distance between shaft centers is 8 ft.
   - (a) Determine the speed of the generator.
   - (b) Determine the angle of contact for each pulley if an open belt is used.
   - (c) Determine the length of belt required.

5. A crossed belt drive is used to connect two shafts. The driven shaft runs at 175 r.p.m. and carries a 28-in. pulley. The shaft centers are 6 ft. apart.

(a) Determine the size of the driven pulley if it is to run at 100 r.p.m.

(b) Determine the angle of contact for each pulley.

(c) Determine the length of belt required.

6. A 20-in. pulley and a 36-in. pulley are used with an open belt to connect two shafts 6 ft. apart. What error is made if the belt length is computed by the approximate formula instead of the exact formula?

7. An 1800 r.p.m. motor is used to drive a main line shaft at 250 r.p.m.

(a) If an 8-in. pulley is used on the motor, what size pulley should be used on the line shaft?

(b) The motor to line shaft center distance is 10 ft., and an open belt is used. What length should the belt be?

8. Two parallel shafts 4 ft. apart are connected by a belt. The pulleys are 15 and 45 in. in diameter.

(a) Calculate the length of belt required if the pulleys revolve in the same direction.

(b) Calculate the length of belt required if the pulleys rotate in opposite directions.

9. A pair of stepped pulleys is used for an open drive with the shafts 42 in. apart. The driving shaft runs at 275 r.p.m., and the smallest driving step is 12 in. in diameter. The driven shaft is to run at 175 r.p.m., 225 r.p.m., and at 300 r.p.m.

(a) Determine the belt length required.

(b) Determine the size of all steps on both step-pulleys.

10. A belt 12 ft. long is used with a pair of step-pulleys. The driving shaft runs at 200 r.p.m. and the driven shaft at 150 r.p.m., at 225 r.p.m., and at 300 r.p.m. The largest pulley on the driven shaft is 20 in. in diameter.

(a) Determine the center distance if an open belt is used.

(b) Determine the size of all steps on both step-pulleys.

11. A lathe is run from a counter shaft by a five-step pulley. The counter shaft rotates at 250 r.p.m., and the lathe is to rotate at 100, 175, 250, 325, and 400 r.p.m. The smallest step on the counter shaft pulley is 10 in. Determine the size of all the steps if a crossed belt is used. The center distance is 8 ft.

12. An engine lathe has on its driving spindle a step-pulley of five steps. The smallest step is 8 in., and the largest step 24 in. in diameter. The counter shaft or driving step-pulley runs at 225 r.p.m.

The slowest speed of the lathe spindle is 75 r.p.m. The five speeds of the lathe are to be in arithmetic series.

Determine the diameter of each step on both step-pulleys if the distance from the lathe spindle to the counter shaft is 6 ft.

13. The step-pulley on a drill-press counter shaft has diameters of 12.5, 9.375, and 7.50 ins., while the step pulley on the drill press drive-shaft has diameters of 8.50, 12.25, and 14.125 ins. Are these step-pulleys designed for an open or crossed belt?

14. Two parallel shafts, 6 ft. apart, are connected by a belt drive so that the driver will turn at 150 r.p.m. and the driven at 50, 150, 325, and 600 r.p.m. The largest and smallest driven pulleys are 30 and 8 in. in diameter. The largest and smallest driving pulleys are 32 and 10 in. in diameter.

(a) If the driver turns clockwise, in what direction does the driven shaft turn?
(b) Determine the remaining pulley sizes.
(c) Would the pulley sizes change if the shafts were moved to 10 ft. centers?

15. A wood lathe is run by an open belt on a three-step pulley. The lathe rotates at 1200 r.p.m. when the belt is on the smallest step, and 300 r.p.m. when on the largest step. What should be the r.p.m. of the counter shaft if equal step-pulleys are used? What is the middle speed of the lathe?

16. Two equal step-pulleys are used with an open belt to give a series of five speeds to a power shaft. The driver runs at 350 r.p.m., and the two highest speeds of the driven shaft are 450 and 397 r.p.m.

(a) Determine the remaining speeds of the driven shaft.
(b) Determine the diameters of all steps if the smallest is 9½ in. and the shafts are 4 ft. apart.

17. A shaft rotating at 300 r.p.m. is to drive a second shaft at five different speeds arranged in a geometric ratio. A crossed belt and equal step-pulleys are to be used.

Determine the diameters of all steps on both pulleys if the lowest speed of the driven shaft is 150 r.p.m.

18. Design a pair of speed-cones for a driving shaft rotating at 200 r.p.m. and a driven shaft having a maximum speed of 150 r.p.m. Use an open belt with the shafts 30 in. apart. The smallest diameter of the driver is to be 10 in., and the speeds are to increase in an arithmetic progression as the belt is moved equal distances along the cones. Total axial movement of belt, 15 in. Make a scale drawing of these cones to ¼ size, indicating dimensions.

19. Using the data of Problem 18, design the speed-cones for a crossed belt.

20. Lay out to scale a pair of cones, using an open-belt drive between shafts, spaced 3 ft. apart. The small end of the driven cone is 8 in. in diameter, the large end is 24 in. in diameter, and the belt movement over the cone is 36 in. The driver rotates at 200 r.p.m., and the highest speed of the driven shaft is 600 r.p.m. As the belt travels along the cones, the driven speed is to change according to an arithmetic progression. Determine five diameters on each cone, and state the corresponding speeds for the driven shaft.

21. Using the data of Problem 18, except that the speeds of the driven are to increase in geometric ratio, design the speed-cones for a crossed belt.

22. On the campus or in near-by commercial plants, locate examples of the use of step-pulleys, speed-cones, tight and loose pulleys, Tex-ropes or other V-belts, short-center drive with a tension pulley, rope-drive, conveyor-chains, roller-chains, and silent-chains.

Describe and illustrate each installation, noting the particular use, main dimensions, and any other interesting features.

23. Sketch a belt drive between two shafts 4 ft. apart and at right angles to each other, using only two pulleys. Indicate the direction of motion of the belt and of the pulleys.

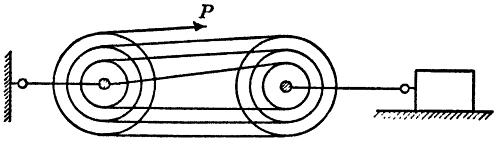
24. Make sketches showing how to connect two shafts, in planes 3 ft. apart, and perpendicular to each other so that the shafts may rotate in both directions. Use a belt with three pulleys and a belt with four pulleys.

25. A belt operates on a 30-in. pulley and an 8-in. pulley with a center distance of 40 in.

(a) Determine the angle of contact on each pulley.

(b) If a 6-in. idler pulley is located with its center 9 in. from the small pulley and on the line of centers, what will be the angle of contact on each of the main pulleys?

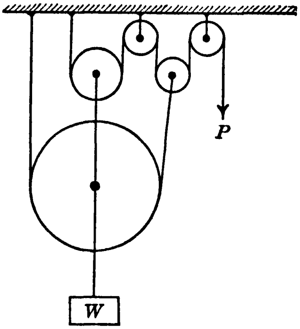
(c) If power transmitted is proportional to the angle of contact, how much has the capacity of the drive been increased in per cent?



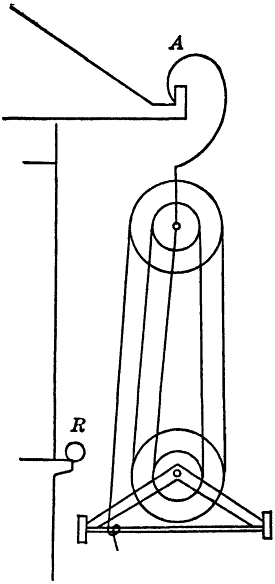
PROBLEM 26

26. In the arrangement of rope blocks shown what is the mechanical advantage? Could this set of blocks be arranged to give better mechanical advantage?

27. Determine the mechanical advantage of the rope arrangement shown in the figure.



PROBLEM 27



PROBLEM 28

28. A painter had his scaffold supported as shown. Seeing a convenient ring at *R*, he tied the loose end of the rope to it. Then the cornice at *A* broke loose, although it supported him with the first arrangement. Prove that this is possible.

29. Make a sketch of each and indicate the difference between the American and English systems of rope drive.

(a) Discuss the advantages of each of the above systems.

(b) Which is more desirable, the rope drive or the belt drive?

30. A rope sheave 18 ft. in diameter, running at 90 r.p.m., drives the main shafts on the second and third floor of a plant by means of ropes.

(a) What will be the r.p.m. of the second floor shaft if a 64-in. sheave is used?

(b) The third floor shaft is to run at 353 r.p.m. What size sheave should be used?

31. In a chain hoist the upper block consists of a 22-pocket chain sheave and a 20-pocket sheave fastened to the same shaft. The lower or hook-block is a 21-pocket chain sheave.

(a) What is the mechanical advantage of this hoist?

(b) If the chain is pulled over the upper sheave at the rate of 50 ft. per min., how fast will the load be lifted?

32. A differential chain hoist is to have a mechanical advantage of 30. The smallest chain sheave should have not less than 17 chain pockets. Determine the number of pockets in each sheave.

33. A 1½-in. pitch chain is used with a 17-tooth sprocket rotating at 150 r.p.m. to drive a sprocket at 50 r.p.m.

(a) How many teeth are there on the driven sprocket?
(b) What is the pitch diameter of each sprocket?
(c) If the center distance is 36 in., what length of the chain is required?

34. A Weston differential pulley block has two sheaves on the upper block, 12 and 11 in. in diameter respectively. Disregarding friction, what weight could be lifted with a pull of 50 lbs.?

35. Determine the ratio of sheave diameters for a differential pulley block which is to lift a 1500-lb. weight with a pull of 75 lbs.

36. A differential pulley block is to lift 2000 lbs. with a pull of 50 lbs. The diameter of the larger upper sheave is 14 in.

(a) Determine the diameter of the smaller upper sheave, disregarding friction.
(b) How many feet of chain must be pulled over to raise the load 8 in.?

37. Make a line sketch of a chain drive and prove that the angular velocity ratio between the driving and driven shafts cannot be constant.

38. A tractor of the caterpillar type has the track made in the form of a chain of 8-in. pitch. The driving sprocket has 15 teeth.

(a) What is the pitch diameter of the sprocket?
(b) When the tractor travels 2½ mi. per hr. what is the speed of the sprocket in r.p.m.?

39. A caterpillar type of tractor has the tracks made of chain links of 10-in. pitch. The driving sprocket has 20 teeth.

(a) What is the pitch diameter of the sprocket?
(b) How many r.p.m. does the sprocket make when the tractor is moving 3 mi. per hr.?

CHAPTER XII

ROLLING CONTACT

174. Rolling Surfaces. A mechanism having a driver which turns a follower by rolling against it without slipping between the contact surfaces is said to have ***pure rolling contact;*** and the surfaces are called ***rolling surfaces.***

When pure rolling contact exists, the point of contact is always on the line of centers, and the arcs of contact between the two surfaces are equal.

Pure rolling contact may be used to connect parallel shafts, intersecting shafts, or shafts in different planes. The driven shaft may have a constant or a variable angular velocity.

175. Parallel Shafts. Two cylinders used to connect parallel shafts by rolling contact constitute a pair of ***friction gears.*** The cylinders may be in either external or internal contact. The angular velocity ratio of two shafts connected by rolling cylinders is inversely as the diameters of the rolling cylinders, and evidently is constant.

176. External Contact. The diameters of the cylinders necessary to connect two shafts with a given velocity ratio can be found mathematically or graphically.

Consider the two shafts *A* and *B*, Fig. 191, 10 inches apart,

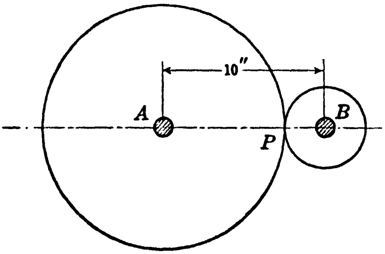


FIG. 191

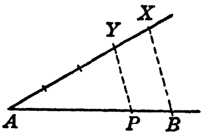


FIG. 192

with A driving and rotating clockwise at 100 r.p.m., and B rotating at 300 r.p.m. We may then write

$$\frac{D_a}{D_b} = \frac{300}{100}, \quad D_a = 3D_b, \quad D_a + D_b = 20,$$

$$4D_b = 20, \quad D_b = 5 \text{ in.}, \quad D_a = 15 \text{ in.}$$

The cylinder diameters can be determined graphically as follows: In Fig. 192, lay off a line AB, equal to twice the center distance, or 20 inches long, and from A draw AX making any angle with AB. From A, lay off AY, three hundred units in length, corresponding to the angular velocity of B. From Y, lay off YX, one hundred units in length, corresponding to the angular velocity of A. Join X and B, and draw YP parallel to XB. Then P is the point of contact between the required cylinders with diameters AP and BP. This construction may be proved as follows. We have

$$AP : PB = AY : YX, \quad \text{but} \quad AY : YX = 300 : 100;$$

therefore

$$\frac{D_a}{D_b} = \frac{300}{100}.$$

177. Internal Contact. When the two shafts turn in the same direction, the rolling cylinders are in internal contact.

Consider the shafts A and B, Fig. 193, 5 inches apart, with A driving and rotating clockwise at 100 r.p.m., and B rotating at 300 r.p.m.

$$\frac{D_a}{D_b} = \frac{300}{100}, \quad D_a - D_b = 5 \times 2 = 10, \quad D_a = 3D_b,$$

$$2D_b = 10, \quad D_b = 5, \quad D_a = 15.$$

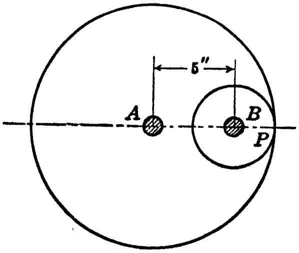


FIG. 193

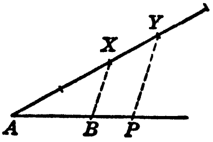


FIG. 194

The cylinder diameters can be determined graphically as follows. Draw a line AB, Fig. 194, 5 inches long, and draw AY, making any angle with AB. Make AY three hundred units, corresponding to the speed of B, and make YX one hundred units, corresponding to the speed of A. Join X and B, and draw a line through Y parallel to XB, cutting AB, produced, at P. Then P is the point of contact between the two required rolling cylinders with BP and AP as radii. This construction may be proved as follows:

$$AP : BP = AY : XY = 300 : 100;$$

therefore

$$\frac{D_a}{D_b} = \frac{R_a}{R_b} = \frac{300}{100}.$$

178. Intersecting Shafts. When the shafts are not parallel but intersect, the rolling surfaces will be cones with their apexes at the point of intersection. The angular velocities of the shafts are inversely proportional to the diameters of the bases of the cones, or to the diameters at any point of tangency. Rolling cones may have either external contact or internal contact. Referring to Fig. 195, we have

$$\frac{N_a}{N_b} = \frac{R_b}{R_a} = \frac{OX \sin \beta}{OX \sin \alpha} = \frac{\sin \beta}{\sin \alpha}.$$

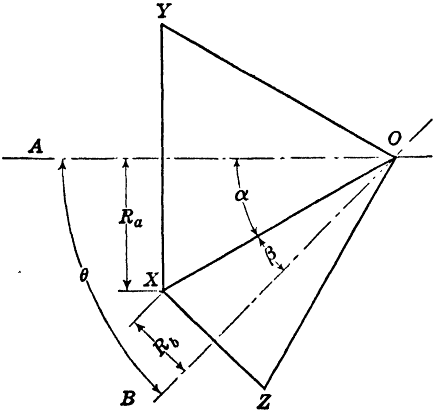


Fig. 195

The cone XOY is the cone on the shaft AO, and XOZ is the cone on the shaft BO. The line OX is the line of contact between the two cones. The angle α is the half angle of the cone XOY, called the ***center angle,*** and the angle β is the center angle of the cone XOZ. The angle θ is the angle between the shafts or ***shaft angle.*** The cone center angle can be found either graphically or by calculation. If the center angle is found by calculation, it is best to express it in terms of the shaft angle and the angular velocities of the two shafts, since these are usually known. The expression for the center angle can be derived as follows, referring to Fig. 195.

$$\frac{N_a}{N_b} = \frac{\sin\beta}{\sin\alpha} = \frac{\sin\beta}{\sin(\theta-\beta)} = \frac{\sin\beta}{\sin\theta\cos\beta - \cos\theta\sin\beta}.$$

Dividing the numerator and denominator by cos β, we find

$$\frac{N_a}{N_b} = \frac{\tan\beta}{\sin\theta - \cos\theta\tan\beta},$$

$$\frac{N_b}{N_a}\tan\beta = \sin\theta - \cos\theta\tan\beta.$$

Hence

$$\text{(87)} \qquad \tan\beta = \frac{\sin\theta}{\frac{N_b}{N_a} + \cos\theta},$$

and, similarly,

$$\text{(88)} \qquad \tan\alpha = \frac{\sin\theta}{\frac{N_a}{N_b} + \cos\theta}.$$

If the cones XOY and XOZ are in internal contact as shown in Fig. 196, the angle $\theta = \alpha - \beta$, and equations (87) and (88) become:

$$\text{(89)} \quad \tan\beta = \frac{\sin\theta}{\frac{N_b}{N_a} - \cos\theta},$$

and

$$\text{(90)} \quad \tan\alpha = \frac{-\sin\theta}{\frac{N_a}{N_b} - \cos\theta}.$$

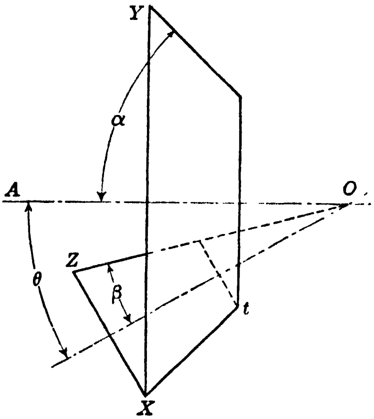


FIG. 196

After the center angles are found, the cones can be drawn in with any desired diameter for the base and length for the face. In Fig. 196, if XY is chosen as the base of the cone XOY, then the base XZ of the cone XOZ is determined. The face of the cone is XT.

179. Graphical Solution for External Contact. The center angles may be found graphically as follows. In Fig. 197, the shafts AO and BO intersect at 60°, and AO drives and rotates at 100 r.p.m. The diameter of the base of the cone on AO is 5 inches, and the face of the cones is $1\frac{1}{2}$ inches. The shaft BO rotates at 200 r.p.m.

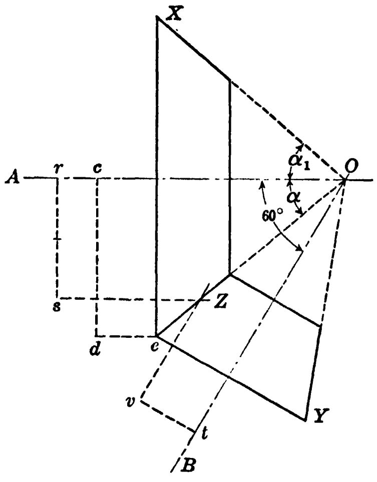


FIG. 197

Draw a line perpendicular to AO through any point r. Lay off rs, two hundred units in length to any convenient scale, to represent the speed of BO. Draw a line perpendicular to BO through any point t. Lay off tv, one hundred units in length, to represent the speed of AO. Draw a line through v parallel to BO, intersecting a line through s parallel to AO at Z. A line through O and Z will be the line of contact of two cones giving the required speed ratio between AO and BO. Make the angle α_1 equal to angle α, locating the line OX. In like manner locate OY. Draw cd perpendicular to AO at any point c, and equal to $2\frac{1}{2}$ inches, the required radius of the base of the cone on AO. Through d draw a line parallel to AO, cutting OZ at e. Then eOX and eOY are the required cones.

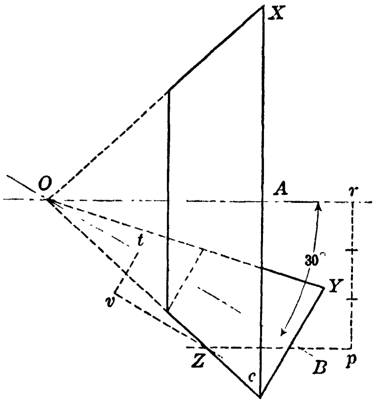


FIG. 198

180. Graphical Solution for Internal Contact. In Fig. 198, the shafts AO and BO intersect at 30°. The shaft AO drives and rotates at 100 r.p.m. The shaft BO rotates at 300 r.p.m.

Draw a line through any point r perpendicular to AO, and from r lay off rp, three hundred units in length, to represent the speed of BO. Draw a line through any point t perpendicular to BO, and from t lay off tv, one hundred units in length, to represent the speed of AO. Through v draw a line parallel to BO, intersecting a line through p parallel to AO at Z. Then the line of contact between the two cones will be along OZ. The cones are completed as in the previous article.

181. Rolling Curves. There are a number of curved surfaces that will transmit motion with pure rolling contact when paired with another curved surface of the same kind, for example, pairs of surfaces whose projections on a plane perpendicular to the axes of rotation form two equal parabolas, two equal hyperbolas, two equal ellipses, or two logarithmic spirals. However, only the ellipses, the hyperboloid of revolution, and a general curve will be discussed here.

182. Equal Ellipses. In Fig. 199, the two equal ellipses, X and Y, are in contact at C. The ellipse X is turning about its focus D, and Y is turning about its focus E. The ellipses are so placed relative to each other that the distance DE between the fixed axis is equal to the major axis of the ellipse. When the ellipses are so placed, the point of contact C will always be on the line of centers DE, and the surfaces will have pure rolling contact. If the ellipse X rotates with a constant angular velocity, Y will rotate with a variable angular velocity, varying with the ratio of DC and EC. The rolling ellipses are used as pitch curves in the design of elliptical gears.

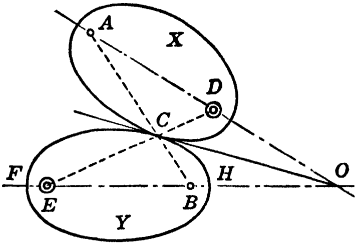


FIG. 199

PROOF. The ellipses X and Y were drawn equal and symmetrical with respect to the common tangent OC. Therefore

$$AC = EC, \quad \text{and} \quad CD = CB;$$

adding these equations, we find

$$AC + CD = EC + CB, \quad EC + CD = EC + CB,$$

but

$$EC + CB = FH = \text{constant};$$

therefore

$$EC + CD = ED = FH = \text{constant}.$$

Since ED is constant, the points E and D may be fixed, and the ellipses X and Y will roll together with pure rolling contact. A linkage with cranks DA and EB, with a connecting link AB, can be substituted for the rolling ellipses X and Y without interfering with the angular speed ratio which exists between the two shafts, D and E.

Since the connecting link AB crosses the line of centers ED at C, we have

$$\frac{\text{angular velocity of } X}{\text{angular velocity of } Y} = \frac{EC}{DC}.$$

For a proof of this result, see § 55.

183. Rolling Hyperboloids. Hyperboloids of revolution may be used as rolling surfaces connecting two shafts which are neither parallel nor intersecting.

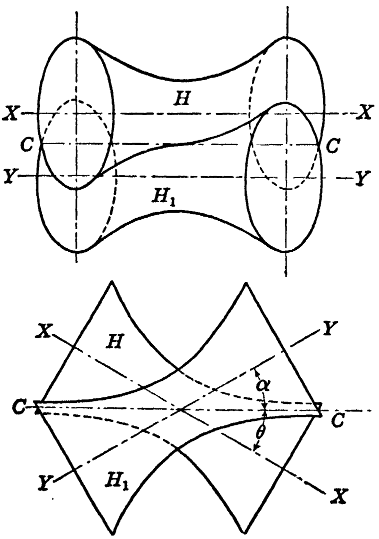


Fig. 200

In Fig. 200, if the straight line CC, not parallel to the axis, is rotated about the axis XX and then about the axis YY, two hyperboloids will be developed. Since the elements of these hyperboloid surfaces are straight lines, the axes can be placed so that the surfaces will be tangent along one pair of these elements. As the hyperboloids turn, they will remain tangent along a straight line, and rolling contact will exist between the hyperboloids on the shafts XX and YY. There will also be some endwise slippage between the elements of the rolling curves.

If XX is the driver, the angular velocity ratio between the two shafts may be found from the expression

$$\frac{N_y}{N_x} = \frac{D_x \sin \theta}{D_y \sin \alpha}. \tag{91}$$

Sections of rolling hyperboloids are used as pitch surfaces for skew bevel gears. Rolling hyperboloids are also used in pairs to serve as ***straightening rolls*** in mills manufacturing iron and steel pipe and round bars.

184. Non-Uniform Velocity-Ratio by Rolling Contact. In some machines, it is desirable to drive a shaft at non-uniform angular velocity from a shaft rotating at constant angular velocity. This is possible when irregular pitch curves of variable diameter are used. In the general case, the distance between shafts is known, and the law of angular velocity of the driven gear is known. The problem then resolves itself into dividing the center distance into the inverse ratio of the angular velocities, at the same time keeping the lengths of the arcs of the two-pitch curves which roll together in any given time interval equal to each other.

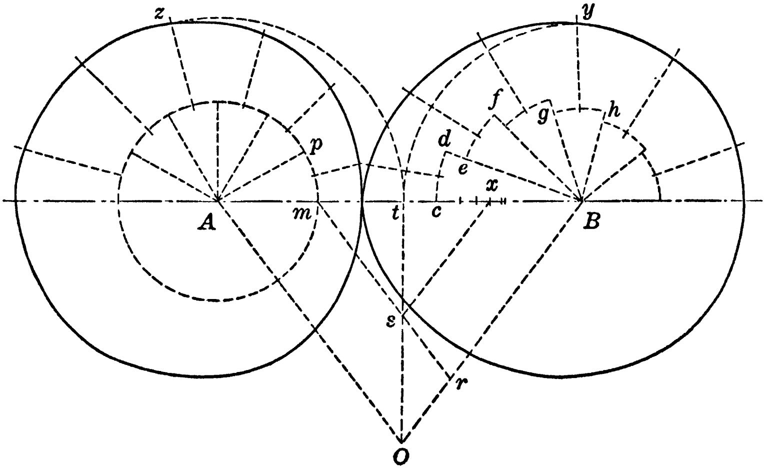


FIG. 201

The construction for the case is given in Fig. 201 and in the following example.

The shaft A is the driver and rotates at a constant angular velocity. The shaft B is the driven shaft and is to rotate according to a definite law of variable angular velocity. To find the required pitch curves, proceed as follows:

Draw circle with radius Am, and divide the first 180°, opposite to rotation, into any number of equal parts, in this case six. On shaft B divide the first 180°, opposite to rotation, into six parts to give the desired angular speed ratio between the two shafts. That is,

$$\frac{\text{angular velocity } B}{\text{angular velocity } A} = \frac{\text{angle passed through on } A}{\text{angle passed through on } B}.$$

Make the arcs cd, ef, and gh equal to pm. To find the point for the fourth division, rotate gh about B, cutting the line of centers AB at x. Draw any lines from A and B, intersecting at O. Draw mr parallel to AO and xs parallel to BO. From O draw a line through s, cutting the line of centers at t. With B as a center and Bt as a radius, cut the bisector of the fourth angle at y, and in like manner find the point z on A. Then y and z will be points on the respective rolling curves, and other points will be found in like manner. The second 180° was laid out with the same variation in angular velocity as the first 180°. Curves drawn through the points found for each shaft will give the correct outline for surfaces which roll together with pure rolling contact. These curves may be used as pitch lines on which to construct tooth curves if a positive drive is desired.

To prove this construction, we proceed as follows. Since arc gh = arc mp, we may write

$$\frac{\text{average angular velocity of } B}{\text{average angular velocity of } A} = \frac{gh}{Bg} \div \frac{mp}{Am} = \frac{Am}{Bg} = \frac{Ap}{Bx}.$$

By construction, we have

$$At : Ap = Ot : Os = Bt : Bx.$$

Therefore

$$\frac{At}{Bt} = \frac{Ap}{Bx} = \frac{\text{average angular velocity of } B}{\text{average angular velocity of } A}.$$

This is the average velocity ratio for the fourth division.

185. Variable Speed Devices. It is possible to vary the speed of the driven shaft, when the driving shaft is rotating at a constant speed, by special mechanisms having rolling contact. The following examples will serve to illustrate some of the applications of rolling contact to variable speed devices.

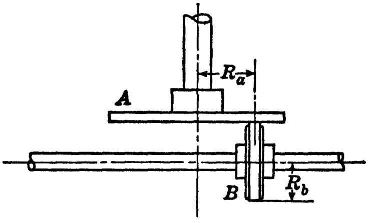


FIG. 202

In Fig. 202, A is a driving disc fixed to its shaft. The driven wheel B is keyed to its shaft but free to move along the shaft. If the disc A rotates at a constant angular velocity, B will rotate with an angular velocity depending upon its position along the shaft, since

$$\frac{N_b}{N_a} = \frac{R_a}{R_b}.$$

The angular velocity of the shaft B, therefore, may be varied by changing the radius R_a. The rotation of shaft B may be reversed if the wheel is moved past the center of rotation of the shaft A.

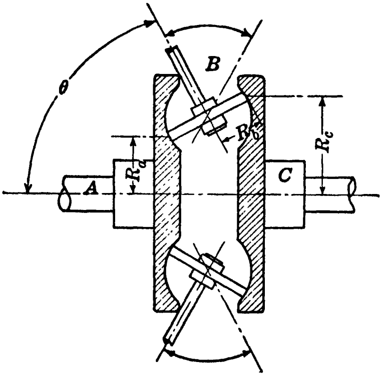


FIG. 203

Figure 203 illustrates another device that may be used to vary the speed of the driven shaft by rolling contact. The shaft A drives with a uniform angular velocity: and the angular velocity of C will vary, depending on the position of the wheel B. The position of the shaft B is changed by some external mechanism, thus changing the angle θ, and the radii R_a and R_c. If no slippage occurs between the rolling surfaces, we have

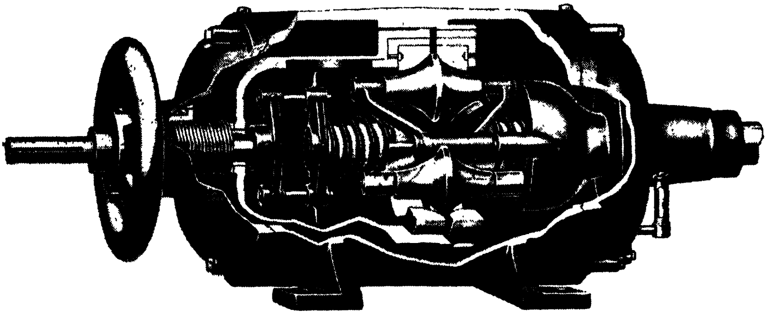
$$N_b = \frac{N_a R_a}{R_b},$$

and

$$N_c = \frac{N_b R_b}{R_c} = \frac{N_a R_a}{R_b} \times \frac{R_b}{R_c} = \frac{N_a R_a}{R_c}.$$

The rotation of shaft C may, therefore, be varied by increasing or decreasing the angle θ.

A commercial device, usually installed between the source of power and the machine to reduce and vary the speed, is shown in Fig. 204. The device is known as the JFS Speed Changer. The high-speed shaft operates, through two adjustable inner races, a system of cone-shaped rollers. These cones also roll on an adjustable outer race which determines the speed of the driven shaft. A detailed discussion of all parts may be found in a pamphlet published by the manufacturers of this device.



Stephens-Adamson Mfg. Co., Aurora, Ill.

FIG. 204. JFS SPEED CHANGER.

PROBLEMS

1. Two parallel shafts, 6 in. apart, are to rotate in opposite directions, and the driver is to rotate twice as fast as the driven. Determine graphically the diameters of the rolling cylinders required. Check your results mathematically.

2. Two parallel shafts 7½ in. apart rotate in opposite directions at 75 and 225 r.p.m. respectively. Determine the diameter of the required rolling cylinders graphically and check your results mathematically.

3. Two parallel shafts rotate in the same direction at 125 and 250 r.p.m., and their center distance is 4 in. Determine the diameters of the required rolling cylinders graphically and check your results mathematically.

4. Two shafts intersect at an angle of 90°. The driving shaft is to rotate at 100 r.p.m., and the driven shaft at 300 r.p.m. The large diameter of the driving cone is 6 in.

   (a) Determine the large diameter of the driven cone.
   (b) Determine the center angle of each cone, graphically.
   (c) Determine the center angle of each cone, mathematically.

5. Two shafts intersecting at 60° are connected by rolling cones having internal contact. The driver rotates at 50 r.p.m. and the driven at 125 r.p.m.

   (a) If the driver is 5 in. in diameter at the large end, what is the diameter of the driven?
   (b) Determine the center angle of each cone mathematically.

6. Three shafts, A, B, and C, are connected by rolling cones. The driving shaft A rotates at 100 r.p.m. and carries a cone whose large diameter is 2 in. Shaft B makes an angle of 90° with A and 60° with C and is between A and C. The shaft B rotates at 200 r.p.m.

   (a) Determine the large diameter of the cones on B and C.
   (b) Determine the speed of B and C.
   (c) Determine the center angle of each cone.

7. A flat disc rotating on its axis is used to drive a cone whose large diameter is 3 in. The cone rotates 3 times as fast as the disc. Determine the diameter of the disc and the angle which the axis of the cone makes with the axis of the disc.

8. A pair of rolling ellipses have centers of rotation 8 in. apart. The angular velocity ratio is to vary from 0.4 to 2.5. Determine distance between the foci, and the major and minor axes.

9. A pair of equal ellipses having major axes of 2 in. and minor axes of 1½ in. are used to transmit motion from a shaft A to a shaft B. Shaft A turns at 25 r.p.m.

   (a) How far apart are the shaft centers?
   (b) What is the maximum angular velocity of B?
   (c) What is the minimum angular velocity of B?

10. Two shafts, A and B, are 4 in. apart. They are to be connected by a pair of rolling ellipses so that the maximum angular velocity ratio will be 2 to 1.

    (a) Determine the major and minor axes of the ellipses.
    (b) Plot a curve showing the variation in angular velocity for one complete revolution of A.

11. Two shafts making an angle of 30° with each other are 3 in. apart at their nearest points. The shafts are to rotate with a velocity ratio of 3 to 2. What kind of rolling surfaces are required? What will be their diameters?

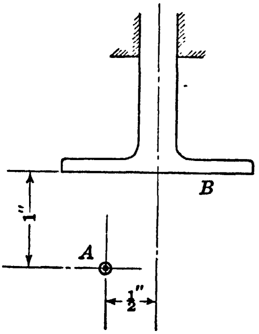
12. Two shafts in parallel planes are not parallel. Their projections on one of these planes intersect at 30°. The shortest distance between the axes is 10 in. If the driver is to rotate at 25 r.p.m. and the driven at 75 r.p.m., determine the small or throat diameters of the required rolling hyperboloids.

13. Two parallel shafts 3 in. apart are to be connected by rolling surfaces so that the driver will rotate at a uniform velocity of 60 r.p.m., and the driven will rotate with uniformly increasing velocity for 180°, uniform velocity for 60° and uniformly decreasing velocity for 120°. The minimum velocity of the driven is to be 60 r.p.m. and the maximum 90 r.p.m. Determine the outlines of the rolling surfaces.

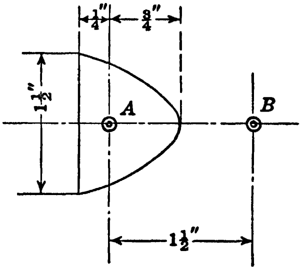
14. A shaft rotating with a uniform angular velocity of 30 radians per minute is to drive a second shaft by means of a pair of rolling surfaces. The second shaft is to rotate with a uniform angular velocity of 15 radians per minute for a quarter revolution. During the next half revolution the driven shaft is to be uniformly accelerated to an angular velocity of 60 radians per minute. During the last quarter revolution the driven shaft is uniformly decelerated to its original angular velocity. Lay out the rolling surfaces if the shafts are $2\frac{1}{2}$ in. apart.

15. Plot a closed curve rotating on a center *B*, which will roll on a 2-in. circle with its center of rotation *A*, offset $\frac{1}{2}$ in. from the center of the circle. Shafts *A* and *B* are 3 in. apart.

16. Plot a curve rotating on a shaft at *A* which will have rolling contact with the surface *B*, raising *B* a distance of 1 in. with uniform velocity in 90° rotation of *A*.



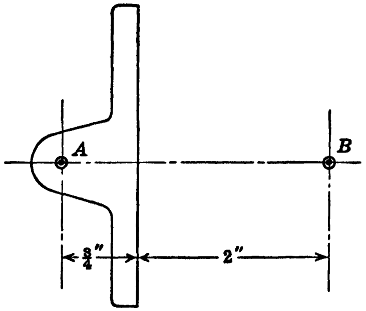
PROBLEM 16



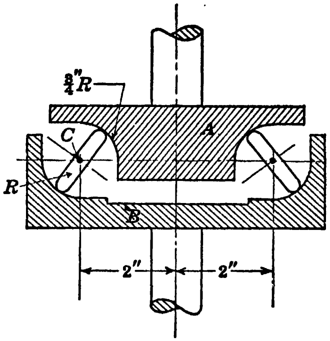
PROBLEM 17

17. Plot a closed curve rotating on the center *B* which will roll on a parabola and straight surface rotating on the shaft *A*. The focus of the parabola is at *A*.

18. Plot a closed curve rotating on the center B which will roll on the straight surface oscillating about the center A.



PROBLEM 18



PROBLEM 19

19. The roller R in the figure is supported in such a way that it can be turned on the center C. The shaft A rotates at 100 r.p.m.

(a) When the axis of the roller makes an angle of 60° with the vertical, what will be the angular velocity of B?

(b) What is the maximum possible angular velocity of B?

(c) What is the minimum possible angular velocity of B?

CHAPTER XIII

GEARS

186. Toothed Gearing. If two cylinders are mounted on parallel axes and their surfaces brought into contact, and one of the cylinders is revolved, then the second will revolve. The power which can be transmitted from one cylinder to the other depends upon the tangential force exerted between the surfaces, and this, in turn, depends on the pressure with which the surfaces are held together, and on the coefficient of friction of the two surfaces. As the power transmitted increases, the cylindrical surfaces may slip, and the velocity ratio of the two axes will not remain constant. To maintain uniform velocity ratio, properly formed projections or teeth are built up on these cylindrical surfaces. These teeth engage and prevent slippage. The teeth themselves do not merely roll together, but slide on each other, the teeth being in reality a series of cams with sliding followers.

These toothed cylinders are known as ***spur gears.*** Instead of cylindrical surfaces, we can use other rolling surfaces on which to form the teeth, in which case we have ***non-circular, bevel,*** or ***hyperbolic gearing.***

A knowledge of the names and terms applied to the different parts of gears is necessary for an intelligent discussion of their properties and limitations. A study of Fig. 205 and the following definitions will bring out the terms used in connection with spur gears.

187. Gearing Definitions. ***Pitch surface*** is the rolling surface upon which the teeth are formed, or the surface which by rolling on the pitch surface of the mating gear will transmit the same velocity ratio as the gear teeth.

Pitch circle is the imaginary line formed by the intersection of the pitch cylinder and a plane perpendicular to the axis of rotation.

Pitch line is the line of intersection of the pitch surface and the tooth surface.

Pitch point is the point of tangency of the pitch circles or the intersection of the line of centers and the pitch circle.

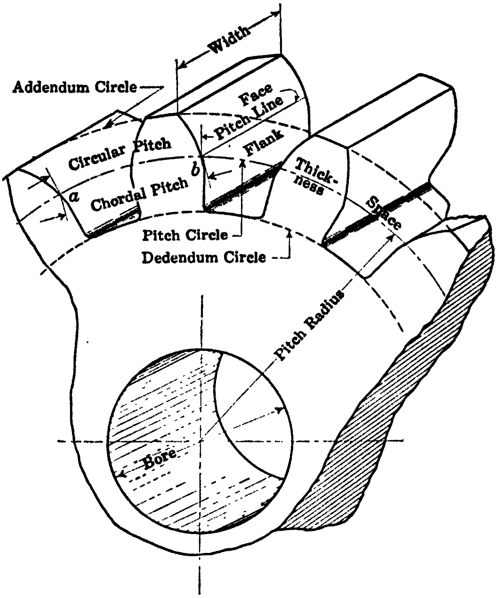


FIG. 205

Line of centers is the line joining the centers of rotation.

Circular pitch is the distance from any point on one tooth to the corresponding point on the next tooth, measured on the pitch circle.

Chordal pitch is the distance from any point on one tooth to the corresponding point on the next tooth, measured as a chord of the pitch circle.

Diametral pitch is the ratio of the number of teeth on the gear to the pitch diameter in inches, or, it is the number of teeth per inch of diameter.

Normal pitch is a term applied to involute teeth and is the distance from one tooth surface to the next tooth surface, measured perpendicular to the tooth surfaces.

Tooth thickness is the thickness of the tooth measured as an arc of the pitch circle.

Tooth space is the distance between two adjacent teeth measured as an arc of the pitch circle.

Backlash is the difference between the tooth space and the tooth thickness measured on the pitch circle.

Addendum is the radial distance from the pitch circle to the top of the teeth.

Dedendum is the radial distance from the pitch circle to the bottom of the space between the teeth.

Addendum circle is the circle forming the tops of the teeth on circular gearing.

Dedendum circle or root circle is the circle touching the bottoms of the spaces between the teeth.

Base circle is an imaginary circle from which the tooth curves of involute gears are generated.

Describing circle, or ***generating circle,*** is an imaginary circle which is rolled on the pitch circles to generate the tooth curves of cycloidal gears.

Tooth face is the tooth surface between the addendum circle and the pitch circle.

Tooth flank is the tooth surface between the pitch circle and the dedendum circle.

Working depth is the distance between the addendum circles of mating gears, measured on the line of centers.

Clearance is the amount by which the dedendum of one gear exceeds the addendum of the mating gear.

Width of face, width of gear, or ***face of gear*** is the length of the tooth measured along an element of the pitch surface.

Point of contact is the point of tangency between the driving and the driven tooth.

Path of contact, or ***line of contact,*** is the path followed by the point of contact of two mating teeth from the time contact begins until contact ceases.

Pressure line is the line joining the point of contact and the pitch point, and is perpendicular to the tooth surfaces at the point of contact.

Pressure angle is the angle between the pressure line and the line drawn perpendicular to the line of centers at the pitch point. This is also called the ***angle of obliquity.***

Angle of action is the angle through which the gear turns from the beginning of contact of one of its teeth with its mating tooth until contact ends.

Angle of approach is the angle through which the gear turns from the beginning of contact of one of its teeth until contact is at the pitch point.

Angle of recess is the angle through which the gear turns from contact at the pitch point until contact ceases.

Arc of approach and ***arc of recess*** are the arcs of the pitch circle subtended by the angles of approach and recess.

188. Comparison of Gear Pitches. The circular pitch of any gear is the circumference of the pitch circle divided by the number of teeth. The circumference is the pitch diameter times π, and therefore is not capable of exact measurement since π equals $3.14159\cdots$ to an infinite number of decimal places. If circular pitch is used in gear teeth measurement, several difficulties are encountered. If an even number or a simple fraction is used as the pitch, the diameters will be inconvenient numbers; or, if the diameters are even numbers, the circular pitch will be an inconvenient number. Computations are clumsy and inaccurate and are a constant source of error. Diametral pitch is easier to use in computations and less liable to error, and is therefore rapidly displacing the circular pitch in all gear work.

Diametral pitch is merely a ratio, namely the number of teeth divided by the pitch diameter in inches. A gear with 20 teeth and a pitch diameter of 10 has a diametral pitch of 2. Since some designers still persist in using circular pitch, the following relations should be noted:

$$(92) \qquad P_d = \frac{T}{D}, \quad p = \frac{\pi D}{T}, \quad p = \frac{\pi}{P_d},$$

where P_d = diametral pitch, p = circular pitch in inches, T = number of teeth, D = pitch diameter in inches.

Chordal pitch is of use in the inspection of gears and in the spacing of the teeth on drawings. Its value can be found by means of the equation

$$\text{(93)} \qquad \text{chordal pitch} = D \sin \frac{180}{T}\,.$$

189. Standard Pitches.* To reduce the infinite number of pitches that might be used to a practical, uniform basis, certain values have been selected as standard pitches. These are shown in the following tables. Table 1 gives the common standard circular and corresponding diametral pitches and Table 2 the diametral and corresponding circular pitch values.

TABLE 1

CIRCULAR PITCH WITH CORRESPONDING DIAMETRAL PITCH

CIRC. PITCH	DIAM. PITCH	CIRC. PITCH	DIAM. PITCH	CIRC. PITCH	DIAM. PITCH	CIRC. PITCH	DIAM. PITCH
p	P_d	p	P_d	p	P_d	p	P_d
2	1.5708	1 5/16	2.3936	15/16	3.3510	9/16	5.5851
1 7/8	1.6755	1 1/4	2.5133	7/8	3.5904	1/2	6.2832
1 3/4	1.7956	1 3/16	2.6456	13/16	3.8666	7/16	7.1808
1 5/8	1.9333	1 1/8	2.7925	3/4	4.1888	3/8	8.3776
1 1/2	2.0944	1 1/16	2.9568	11/16	4.5696	5/16	10.0531
1 7/16	2.1855	1	3.1416	5/8	5.0265	1/4	12.5664
1 3/8	2.2848					3/16	16.7552

TABLE 2

DIAMETRAL PITCH WITH CORRESPONDING CIRCULAR PITCH

DIAM. PITCH	CIRC. PITCH	DIAM. PITCH	CIRC. PITCH	DIAM. PITCH	CIRC. PITCH	DIAM. PITCH	CIRC. PITCH
P_d	p	P_d	p	P_d	p	P_d	p
1/2	6.2832	2	1.5708	4	.7854	10	.3142
3/4	4.1888	2 1/4	1.3963	5	.6283	11	.2856
1	3.1416	2 1/2	1.2566	6	.5236	12	.2618
1 1/4	2.5133	2 3/4	1.1424	7	.4488	13	.2417
1 1/2	2.0944	3	1.0472	8	.3927	14	.2244
1 3/4	1.7952	3 1/2	.8976	9	.3491	15	.2094
						16	.1963

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* These standards have been generally adopted by gear manufacturers.

190. Required Shape of Spur Gear Teeth. In circular gearing, the teeth must be of such shape that the velocity ratio of the two mating gears remains constant. To keep the velocity ratio constant, the *line drawn through the point of contact of two mating teeth perpendicular to the tooth curve must always intersect the line of centers at the same point, namely the pitch point.* Stated in a different way, the *common normal to the tooth curves at the point of contact must pass through the pitch point.* This is called the fundamental or ***first law of gearing.***

Other laws or conditions necessary for correct continuous tooth action are as follows.

The normals to the tooth curves must not intersect between the tooth curves and the pitch circle.

The tooth curves must continually remain tangent to each other.

191. Proof of First Law. In Fig. 206, the linear velocity of the point c on gear A, perpendicular to the tooth curve, must equal the linear velocity of the point c on gear B, in the same direction. If this is not true, the teeth will separate.

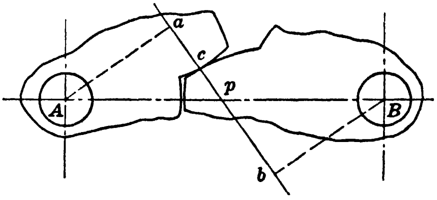

FIG. 206

Draw acb perpendicular to the tooth curve at c, and let ω_a and ω_b be the angular velocities of A and B, respectively. The linear velocity of c on A along ab equals $Aa\omega_a$, or the linear velocity of a. The linear velocity of c on B along ab equals $Bb\omega_b$. These linear velocities must be equal; therefore

$$Aa \times \omega_a = Bb \times \omega_b,$$

and

$$\frac{\omega_a}{\omega_b} = \frac{Bb}{Aa} = \frac{Bp}{Ap} = \text{velocity ratio of } A \text{ and } B.$$

Therefore if at all times, as the point of contact moves along the tooth curves, the normal acb passes through the same pitch point p, the velocity ratio will remain constant.

192. Conjugate Tooth Curves. Given the tooth curve of one gear, to draw the tooth curve of a mating gear.

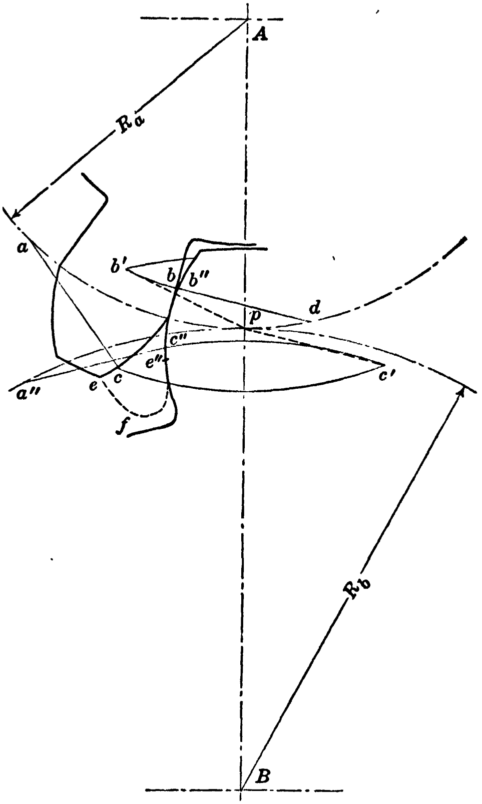

FIG. 207

In Fig. 207, assume the tooth curve *bc* of the gear *A* to be given. Assume any point *c* on the given tooth curve. Draw *ca* perpendicular to the tooth curve. The problem is to find a point on the tooth curve of the gear *B* which will come into contact with *c* when *a* is at the pitch point. Revolve *ac* about the center *A* until *a* falls on *p;* then *c* will be at *c'* and must be in contact with a point *c'* on the gear *B*. Revolve *pc'* about the center *B* to the position *a''c''* so that the arcs *ap* and *a''p* are equal. Then *c''* is a point on the tooth curve of the gear *B*. In the same way *b''* is a point on the tooth curve of gear *B* which will come into contact with the point *b* on the tooth curve of the gear *A*. As many points as are required to determine the tooth curve of the gear *B* can be found in the same way.

By similar construction, *e''* is found to be the point with which the corner *e* of the tooth *A* comes into contact. Only the part of the tooth surface between *e''* and the pitch line is used to transmit motion or power. However, as the addendum of the tooth on *A* extends beyond *e''*, the tooth space on *B* must be cut away for clearance. As the gears rotate, the point *e* will trace a curve *efe''* on the gear *B*. Any clearance curve drawn inside of this curve will allow the gears to operate.

193. Generation of Tooth Curves. If any curve, as *G*, Fig. 208, is rolled on a pitch circle *apb*, a tracing point *t* on this curve

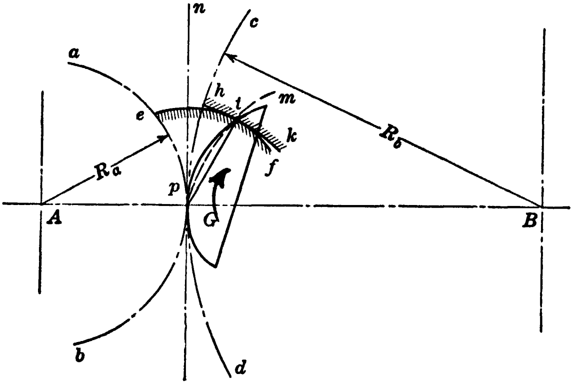

FIG. 208

traces a curve *etf*. The line *tp* drawn from the tracing point to the point of contact between the curve *G* and the pitch line is normal to the curve *etf*. If the same curve *G* is rolled on the inside of the pitch line *cpd*, it will trace a curve *htk* whose normal at *t* is also *tp*. As the position shown is general, it follows that the two curves *etf* and *htk* fulfill the requirements for tooth curves stated in § 190, *etf* being the face of a tooth on *apd*, and *htk* being the flank of a tooth on *cpd*.

Any form of curve can be used for the generating curve *G*, to form the face of one gear and the flank of the mating gear. The same or some other curve can be used to generate the other face and flank. In ordinary gear practice, only two forms of rolling curves are used, the circle for cycloidal gears, and the logarithmic spiral for involute gears.

194. Path of Contact. As the tooth curves *etf* and *htk*, Fig. 208, work together, the point of contact will move along a line *ptm* called the path of contact. This path has a definite relation to the rolling curve. In cycloidal gearing, the path of contact is the rolling or describing circle itself when at the pitch point (see § 199). In involute gearing, it is a straight line (see § 210).

195. Obliquity of Action. When two mating tooth curves are in contact, the force or pressure exerted between them is along the common normal, and therefore passes through the pitch point. As the point of contact moves along the teeth, the angle which the normal, or pressure line, makes with the line drawn normal to the line of centers at the pitch point varies. This angle *tpn*, Fig. 209, is called the angle of obliquity, or pressure angle. In a few forms of teeth this angle is constant.

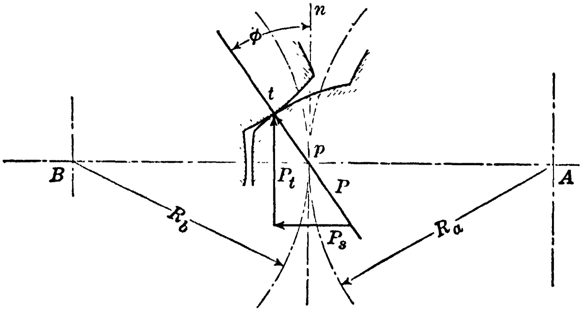

FIG. 209

This obliquity of pressure is of considerable importance. The useful work of turning the gear is done by the component P_t, or the pressure normal to the line of centers. The component P_s parallel to the line of centers tends to push the gears apart. From Fig. 209, it is evident that for a given turning force, P_t, the actual pressure between the teeth, P, increases with the obliquity. As the pressure P increases, the friction between the teeth increases, and consequently the tooth wear increases. It is thus possible that with large angles of obliquity, friction forces will become large enough to cause the gears to lock.

196. Sliding Action of Gear Teeth. The tooth surfaces of gear teeth are designed to give rolling contact between the pitch surfaces, but the teeth themselves have a sliding action. This sliding of the teeth causes friction losses between the tooth surfaces and therefore decreases the efficiency of the gears. The amount of slipping and the velocity of slipping of the tooth surfaces depend on the shape of the teeth and on the distance of the point of contact from the pitch surface.

The amount of slipping can best be studied graphically. In Fig. 210 are shown two gears in mesh with the tooth curves properly designed as previously described. Consider the driving gear to be rotated through equal angles or equal arcs as *p–1*, *1–2*, *2–3*, etc. Then the points of contact of the tooth surfaces can be

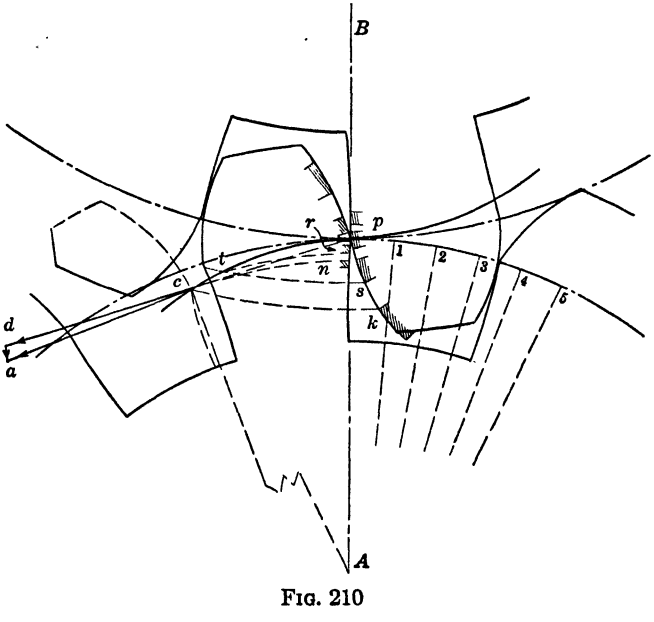

FIG. 210

found when points *1*, *2*, *3*, etc. are at the pitch point *p*. For example, when point *4* is at the pitch point, contact will be at *c*. Draw an arc through *c* about the center of gear *A*, cutting the tooth curve on *A* at *n*. Draw an arc through *c* about the center of gear *B*, cutting the tooth curve on *B* at *k*. Points *n* and *k* are points on the tooth curves which will be in contact at *c* when point *4* is at *p*. In the same way *r* and *s* are points on the tooth curves which will be in contact at *t* when point *3* is at *p*. The length of the tooth curve from *r* to *n* is not equal to the curve from *s* to *k* with which it acts, indicating the sliding action. In the figure, the portions of the tooth curves which slide together when the driver turns through equal arcs on the pitch circle are indicated by the shaded and blank spaces.

To find the actual sliding velocity at any time, consider the gear teeth to be in contact at *c*. The velocity of *c* can be calculated and represented by the vector *ca*, perpendicular to the radial line *Ac*. The line *cpd* is normal to the tooth curves at the point of contact, and the motion of the point of contact on each tooth curve must be equal along this line; otherwise the tooth curves would separate. Resolve the vector *ca* into components along and perpendicular to *cp*, as *cd* and *da*. The vector *da* represents the velocity with which the tooth curves are sliding over each other at the particular instant. Note that when contact is at the pitch point, this velocity is zero, showing no sliding, and that the sliding velocity increases as the point of contact moves on the tooth curve away from the pitch line. This indicates that short teeth have less wear and less power loss by friction than long teeth having the same tooth outline.

PROBLEMS

1. A gear has 30 teeth of 2-in. circular pitch. Determine the pitch diameter, the diametral pitch, and the chordal pitch.

2. A gear has 24 teeth of 3 diametral pitch. What is the pitch diameter, circular pitch, and the chordal pitch?

3. Gear *A* is 12 in. in diameter and has teeth of 4 diametral pitch; it rotates at 300 r.p.m. and drives gear *B* at 200 r.p.m.

   (a) What is the pitch diameter of gear *B*?
   (b) How many teeth has gear *B*?

4. Two cylinders, *C* and *D*, rotating at 40 and 60 r.p.m., are in rolling contact, and *C* is 6 in. in diameter. Teeth are added to the cylinders to make a pair of gears with a circular pitch of 1.0472 in. on *C*. Determine: (a) the number of teeth on *C*, (b) the number of teeth on *D*, (c) the circular pitch of *D*, (d) the diametral pitch of *D*.

5. Convert the following diametral pitches to circular pitches: 1, 1¼, 1.572, 0.7854, and 3.

6. Convert the following circular pitches to diametral pitches: 1 in., 1¼ in., 2½ in., 6.283 in., and 4.712 in.

7. Two shafts 10 in. apart are connected by spur gears giving the shafts an angular velocity ratio of 3 to 2. If the gears have 3 diametral pitch, how many teeth has each?

8. Two shafts 8 in. apart have an angular velocity ratio of 3 to 1 and are connected by gears, the smallest of which has 12 teeth.

(a) What are the pitch diameters of the gears?
(b) What is the number of teeth on the second gear?
(c) What is the circular pitch?

9. Two gears, *A* and *B*, are supported on parallel shafts 20 in. apart. Gear *A* is rotated at 200 r.p.m. and is to drive gear *B* at 50 r.p.m.

(a) What is the diameter of gear *B*?
(b) If *A* rotates clockwise, in what direction does *B* rotate?
(c) If *A* has 20 teeth, determine the circular pitch, the diametral pitch, and the chordal pitch.

10. Two shafts, *A* and *B*, are parallel and are 6 in. apart. *A* turns clockwise and drives *B* counter-clockwise at half the angular velocity of *A*.

(a) Determine graphically the pitch diameters of *A* and *B* and check by calculations.
(b) If the gear found for *A* is retained and the gear on *B* changed so that *B* rotates clockwise, determine graphically the pitch diameters, and check by calculations.

11. A 4-in. gear *A* and a 3-in. gear *B* are fixed on shafts 3½ in. apart. The path of contact is along a circle of 1 in. radius with the center inside the gear *A* and on the center line, and along a circle of ¾-in. radius with its center inside the gear *B* and on the center line. The addendum of each gear is ⅜ in.

(a) If *A* is the driver, determine the maximum and minimum pressure angles.
(b) If *B* is the driver, determine the maximum and minimum pressure angles.

12. The flank of a gear tooth is made of a circle arc of 3-in. radius whose center is 1.05 in. outside of the pitch circle. The face is an arc whose radius is 1.10 in. and whose center is 0.05 in. inside of the pitch circle. The pitch diameter is 9 in. This gear mates with a gear of 6 in. pitch diameter.

Determine points on the mating gear tooth which will mate with points 1/8 and 1/4 in. outside of the pitch circle of the first gear and points 1/8 and 3/16 in. inside of the pitch circle.

13. Using the same tooth curve as given in Problem 12, determine the points on the tooth curve of a rack to mate with the original gear.

14. A pinion of 4 in. pitch diameter drives an annular gear of 10 in. pitch diameter. The flank of the annular is a circle arc of 1¼-in. radius with the center 1/8 in. inside of the pitch circle. Find two points on the face of the pinion, starting with points on the flank of the annular, 3/8 and 3/4 in. from the pitch circle. Show the path of contact in recess.

15. Two equal mating spur-gears are fixed on shafts 6 in. apart. The flank curves are made by circle arcs of 3-in. radius with centers 1.05 in. outside of the pitch circles. The face curves are made by circle arcs of 1.10-in. radius with centers 0.05 in. inside the pitch circles. The addenda are each 3/8 in. Divide the face and flank of one gear into 10 equal divisions and locate the mating points on the other gear. Plot a curve with the 10 equal divisions as ordinates and the corresponding spaces on the mating gear as abscissae. Plot to four times full size.

CHAPTER XIV

CYCLOIDAL GEARS

197. Cycloidal Teeth. In the cycloidal system of gears, the generating curves discussed in § 193 are circles rolling on the outside and the inside of the pitch circles. When the generating circle rolls on the outside of the pitch circle, the tooth curve is an epicycloid; and when the generating circle is rolled on the inside, the tooth curve is a hypocycloid.

This system of gearing is now going out of general use, its place being taken by the involute system. Many cycloidal gears are still in use, however, and the theory of this type of gearing should be thoroughly understood.

198. Effect of Increasing the Describing Circle Diameter. In Fig. 211, if the circle with center at B and radius Bp is rolled to the left, the hypocycloid pb is generated. If the describing circle radius is increased to Cp, the hypocycloid pc is generated. Increasing the describing circle radius brings the hypocycloid closer to the pitch circle diameter pAd, and when the describing circle radius is equal to one-half the pitch circle radius, the hypocycloid is a straight diametral line pAd. Increasing the describing circle radius still more causes the hypocycloid to fall to the right of the diameter. For instance, the describing circle of radius Ep generates the hypocycloid pe.

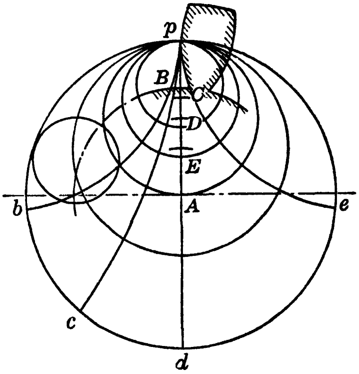

FIG. 211

Since these curves may be used as the flanks of gear teeth, it is evident that making the diameter of the describing circle larger than the radius of the pitch circle will give under-cut teeth as shown. These under-cut teeth are objectionable, since they cannot be readily machined and are extremely weak because of the small root thickness.

Any *hypocycloid can be generated by each of two circles the sum of whose diameters equals the diameter of the pitch circle.* Also any *epicycloid can be generated by each of two circles the difference of whose diameters equals the diameter of the pitch circle.*

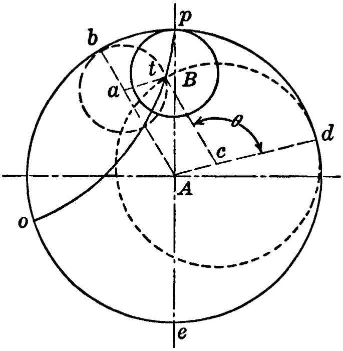

Fig. 212

Proof. In Fig. 212, let the describing circle of radius Bp roll to the left, inside of the circle of radius Ap, generating the hypocycloid *pto.* When B moves to a, the arc bt equals the arc bp. Draw radius at, and draw Ad parallel to at. Make Ac equal to at. Then ct equals Aa equals cd, and the angles bat, aAc, and tcd are equal. Call this angle θ. With c as a center, draw a circle with radius ct. Then we have

$$\text{arc } bpd = \text{arc } bp + \text{arc } pd = \theta Ab = \theta(Aa + ab),$$

and

$$\text{arc } bt + \text{arc } td = \theta ab + \theta ct = \theta(ab + Aa).$$

Therefore

$$\text{arc } bp + \text{arc } pd = \text{arc } bt + \text{arc } td,$$

but

$$\text{arc } bp = \text{arc } bt\,;$$

therefore

$$\text{arc } pd = \text{arc } td.$$

It follows that the point t is a point on the hypocycloid generated by the circle of radius cd. Since cd equals $Ap - ab$, it follows that two circles, the sum of whose radii is equal to the radius of the pitch circle, will generate the same hypocycloid, or cycloidal tooth-flank.

The proof for the epicycloid is similar to that for the hypocycloid.

199. Cycloidal Spur Gears. In Fig. 213, A and B are the centers of two pitch circles in contact at p. The describing circle G_a rolled on the pitch circle of B generates the epicycloid pb.

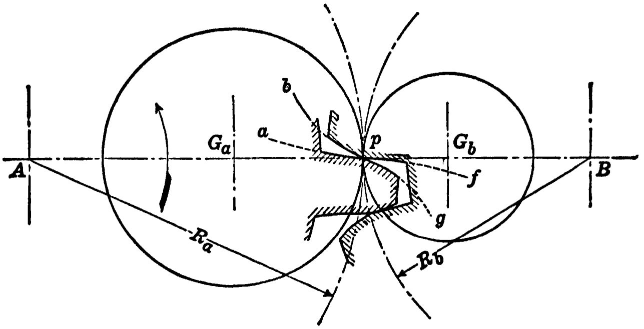

FIG. 213

When rolled on the inside of the pitch circle of A, the circle G_a generates the hypocycloid pa. These curves are the face of the tooth on gear B and the flank of the tooth on gear A.

Similarly, the describing circle G_b generates the curves pf and pg, the curves forming the flank of B and the face of A.

If the centers of A, B, G_a, and G_b are fixed, and A and B rotate together, the two curves will be generated at the same time. If, in Fig. 214, the arc pr is equal to the arc ps, then by construction the point c is a point on both curves or the point of contact of the tooth curves. The line cp is common to both curves and by construction perpendicular to both; and since it always passes through the pitch point p, these curves will always give a constant velocity ratio. As the pitch circles roll together, it is evident that the point of contact c is always on the describing circle, and the describing circle is the path of contact. It is also evident that if two cycloidal gears are to work together, the face of one gear tooth and the flank of the mating gear tooth must be generated by the same describing circle.

200. The Angles of Approach and Recess. If the tops of the teeth in Fig. 214 are formed by the addendum circles shown, the first point of contact will be at the point e, where the addendum of B cuts the path of contact, or the describing circle G_a. Contact will end at *c* where the addendum circle of *A* cuts the path of contact, or the describing circle G_b. When contact is at *e*, the tooth curve crosses the pitch circle at *h*. When the gear rotates until contact is at the pitch point, *h* moves to *p*. Therefore, the angle *hAp* is the ***angle of approach,*** and the arc *hp* is the ***arc of approach*** for the gear *A*.

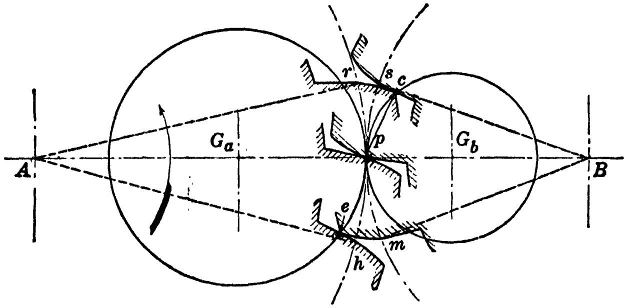

FIG. 214

Similarly, the angle *mBp* is the angle of approach, and the arc *mp* is the arc of approach for the gear *B*. The arcs *hp* and *mp* are equal, but the angles *hAp* and *mBp* are not equal. The angles of recess can be found in the same way.

201. To Determine the Possibility of Correct Action between Two Given Cycloidal Gears. First, the flanks of one gear and the faces of the mating gear must be generated by the same describing circle. Second, the arc of contact must be greater than the circular pitch. Third, the pitches must be the same. Fourth, the teeth must not be pointed below the required addendum.

In Fig. 215, contact will begin at *c* where the addendum of *B* cuts the describing circle G_a. Contact will end at *e*, where the addendum of *A* cuts the describing circle G_b. If the path *cpe* is greater than the circular pitch, these gears will work, provided the teeth are not pointed below the addendum lines.

To determine whether the teeth are pointed, make the arc *pb* equal the arc *pc*. Then *c* and *b* are points on the tooth face curve of the gear *B*. Draw *cB*. If the arc *bd* is less than one-fourth of the circular pitch, or one-half of the tooth thickness, the teeth will not be pointed, and the gears as drawn will work. *If the arc bd is greater than one-fourth the pitch, the teeth will be pointed below the assumed addendum circle, and the gears will not work.*

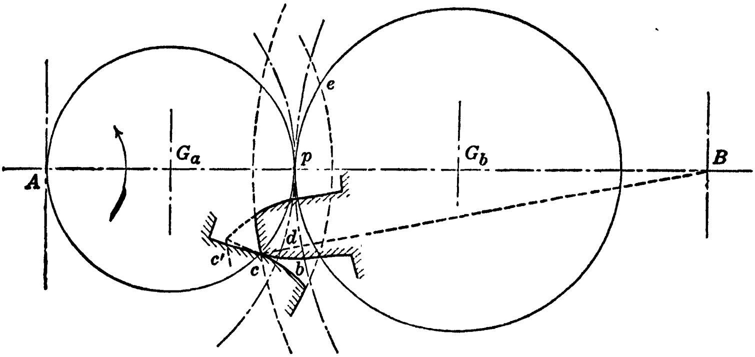

FIG. 215

If a maximum arc of approach is desired, the addendum may be increased until the teeth become pointed, in which case contact will begin at c', Fig. 215.

202. Smallest Gear Mating with a Given Cycloidal Gear. When one gear, as A, in Fig. 215, is given, the smallest mating gear is found by the following construction. One limiting condition is that one pair of teeth must come into contact before the preceding pair has ceased contact. Since the path of contact is equal to the arc of action, it follows that the minimum length of the line of contact is equal to the circular pitch. The path of contact is known since it follows the circumferences of the generating circles of the given gear. One end of the path of contact is at e where the addendum circle of the gear A cuts the generating circle G_b. Lay off from e along the generating circles a distance epc equal to the circular pitch. The addendum circle of the gear B must pass through c, or between c and the center of A.

The second limiting condition is that the pitch circle of B must be greater than the diameter of the describing circle G_b plus the addendum of the gear A. If this is not true, the teeth on B

will have no thickness of metal at the root circle, the flank curves of the two sides of the tooth meeting above the root circle.

The third limiting condition is that the teeth on *B* may be pointed.

Usually after the center of *B* is located, it will be found that the circumference divided by the circular pitch is not a whole number, and the diameter must then be increased to obtain a practical gear.

203. Small Cycloidal Gears. These gears are not of enough importance to require a detailed study. The mathematics of the limiting conditions are involved and must in general be solved by trial and error methods. Each individual case can be tested graphically for possible correct action. For reference, the following cases are given.

A pair of 4-tooth equal gears with pointed teeth will work when the describing circle diameter is $5/8$ of the pitch diameter, but the arc of action is just equal to the pitch, a condition which is undesirable.

A pair of 3-tooth equal gears cannot be made to work.

If the teeth are not to be under-cut, that is, if the describing circle diameter cannot be greater than one-half the pitch circle diameter, the smallest pair of equal gears will have 5 teeth.

For general power transmission, gears with less than 12 teeth with radial flanks should be avoided.

204. Standard Interchangeable Gears. An interchangeable gear set is one in which every gear of any number of teeth will work with every other gear of the same pitch. This condition requires that the face and flank curves must all be generated by describing circles of the same size. In order that gears made by different manufacturers will work together, the describing circle has been standardized so that all twelve tooth gears will have radial flanks. The describing circle then has a diameter equal to the radius of the pitch circle of a twelve-tooth gear. Thus all interchangeable gears of 1 diametral pitch have a 6-inch describing circle, all gears of 2 diametral pitch have a 3-inch describing circle, and so on. A few manufacturers use pinions of 15 teeth with radial flanks as the basis of their interchangeable gear sets. This gives a generating circle diameter equal to ⅝ of the pitch diameter of a 12-tooth pinion.

In order that the maximum pressure angle will not be excessive, the addendum of the teeth has also been limited. Standard gear-tooth proportions are listed in Table 3, in terms of the circular and diametral pitch. These are known as the Brown and Sharpe Standards.

TABLE 3

BROWN AND SHARPE STANDARD GEAR-TOOTH PROPORTIONS

	CAST TEETH	MACHINE CUT TEETH
Circular pitch	p	$p = \frac{\pi}{P_d}$
Addendum	$0.3p$	$0.3183p = \frac{1}{P_d}$
Dedendum	$0.4p$	$0.3683p = \frac{1.16}{P_d}$
Working depth	$0.6p$	$0.6366p = \frac{2}{P_d}$
Total height	$0.7p$	$0.6866p = \frac{2.16}{P_d}$
Tooth thickness	$0.475p$	$0.5000p = \frac{1.57}{P_d}$
Tooth space	$0.525p$	$0.5000p = \frac{1.57}{P_d}$
Clearance	$0.1p$	$0.05p = \frac{.16}{P_d}$
Backlash	$0.05p$	$0 = 0$

205. Drawing Standard Interchangeable Gear Tooth Curves. In detailed drawings of gear teeth, it is seldom necessary to lay out the true cycloidal curves. If the teeth are once laid out to an enlarged scale, the curves may be approximated by two circular arcs, one for the face and one for the flank. Each circular arc passes through three points on the curve, namely the point at the addendum (or dedendum), the pitch point, and the midpoint. The radii for these circular arcs and the location of the arc centers can be found for gears of different numbers of teeth, and tabulated for future use. These values as determined by

Mr. G. B. Grant are tabulated in Table 4, known as Grant's Odontograph Table.

TABLE 4

GRANT'S ODONTOGRAPH FOR CYCLOIDAL GEARS

NUMBER OF TEETH IN THE GEAR		FOR DIAMETRAL PITCH = 1				FOR CIRCULAR PITCH = 1 IN.			
		For Any Other Pitch, Divide by That Pitch				For Any Other Pitch, Multiply by That Pitch			
Exact	Intervals	Faces		Flanks		Faces		Flanks	
		Radius	Distance	Radius	Distance	Radius	Distance	Radius	Distance
10	10	1.99	.02	− 8.00	4.00	.62	.01	−2.55	1.27
11	11	2.00	.04	−11.05	6.50	.63	.01	−3.34	2.07
12	12	2.01	.06	∞		.64	.02	∞	
13½	13–14	2.04	.07	15.10	9.43	.65	.02	4.80	3.00
15½	15–16	2.10	.09	7.86	3.46	.67	.03	2.50	1.10
17½	17–18	2.14	.11	6.13	2.20	.68	.04	1.95	.70
20	19–21	2.20	.13	5.12	1.47	.70	.04	1.63	.50
23	22–24	2.26	.15	4.50	1.13	.72	.05	1.43	.36
27	25–29	2.33	.16	4.10	.96	.74	.05	1.30	.29
33	30–36	2.40	.19	3.80	.72	.76	.06	1.20	.23
42	37–48	2.48	.22	3.52	.63	.79	.07	1.12	.20
58	49–72	2.60	.25	3.33	.54	.83	.08	1.06	.17
97	73–144	2.83	.28	3.14	.44	.90	.09	1.00	.14
290	145–300	2.92	.31	3.00	.38	.93	.10	.95	.12
	Rack	2.96	.34	2.96	.34	.94	.11	.94	.11

To use the odontograph table, draw the pitch, addendum, and dedendum circles. Draw a circle with a radius equal to the pitch radius minus the *face distance.* With a center on this circle and a radius equal to the *face radius,* draw the face curve. Draw a circle with a radius equal to the pitch radius plus the ***flank distance,*** and with a center on this circle and a radius equal to the ***flank radius,*** draw the flank curve.

206. Clearance Curves. As the gears work together, the top point of the tooth face of one gear will trace a curve on the other gear as shown at *ef* in Fig. 207. This curve naturally varies with the size of the gear, and in an interchangeable gear-set the curve of the tooth at the root must be such that it will clear the top of the tooth on every mating gear from the smallest gear to a rack. This clearance is usually provided for by making the curve joining the root circle and the tooth flank into a fillet, or circle arc, of radius equal to one-seventh of the distance across the tooth space at the addendum circle.

207. Racks. If the diameter of the pitch circle is increased to infinity, the pitch circle becomes a straight line, and the gear is called a rack. The rack has all the properties of a spur gear.

208. Annular Gears. If the pitch circle of one gear is contained inside the pitch circle of the mating gear, the larger has its teeth on the inside and is called an annular gear. The action of annular gears is similar to that of spur gears except for the interference noted in the following article. With the annular and pinion, both gears rotate in the same direction instead of in opposite directions as in spur gears.

209. Interference in Annular Gears. In § 198, it is shown that the cycloidal curves, used for the flank and face curves, can each be generated by two distinct describing circles. This fact causes an interesting and important case of interference in cycloidal annular gearing.

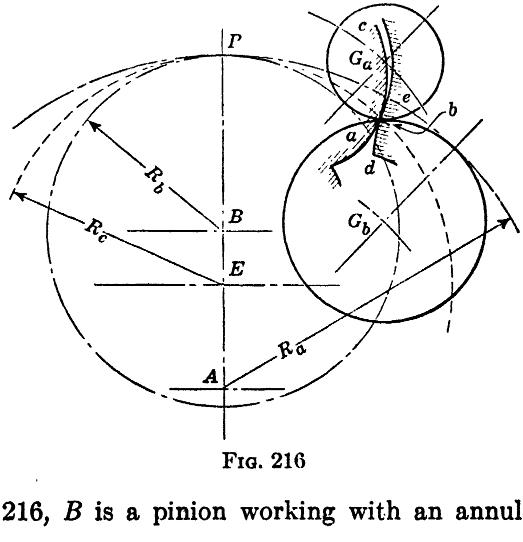

FIG. 216

In Fig. 216, B is a pinion working with an annular gear A.

Choose describing circles G_a and G_b such that the sum of the describing circle diameters equals the difference of the gear pitch diameters. The circle G_b rolling inside the annular pitch line generates the annular face curve *ebd*. This same curve is also generated when the circle *E* is rolled on the inside of the annular pitch line. The circle G_a rolling on the outside of the pinion pitch line generates the pinion face curve *abc*. This same curve is generated by the circle *E* rolling on the outside of the pinion pitch line.

Since the curves *abc* and *ebd* are generated by the same describing circle *E*, they will give perfect tooth action with contact at *b* on the circle *E*. The point *b* is a point on the face of the annular and on the face of the pinion. It is evident then that we have face-to-face contact as well as the regular face-to-flank contact.

If the diameter of either describing circle is increased, the face curves will cross, and the gears will not work because of interference.

If the diameter of either describing circle is decreased, the face curves will not be tangent or crossed. The teeth lose this secondary contact, but the gears will work since they still have the regular face-to-flank contact.

It is evident that, in order for a pinion and annular to work without interference, the difference of the pitch diameters must be equal to or greater than the sum of the diameters of the describing circles. With standard interchangeable gears, this is equivalent to saying that the difference in the number of teeth on the annular and pinion must be 12 or more. If the interchangeable system uses radial flanks on other than a 12-tooth pinion, the difference of the numbers of teeth on the annular and pinion must be equal to or greater than the number of teeth on the pinion having radial flanks.

PROBLEMS

1. Using a pitch circle of 4 in. diameter and a generating circle of 1½ in. diameter, construct a hypocycloid and an epicycloid for one complete revolution of the generating circle.

2. Prove that any epicycloid can be generated by each of two circles, the difference of whose diameters is equal to the diameter of the pitch circle.

3. Lay out accurately the tooth curves for a pair of mating gears with 16- and 10-in. pitch diameters. The addendum circles are 18 and 13 in. in diameter respectively, and the root diameters $12\frac{11}{16}$ and $7\frac{11}{16}$ in. There are 16 teeth on the large gear. The generating circles are to be 4 in. in diameter.

4. A cycloidal gear with a 7-in. pitch diameter has its tooth faces generated by a 2-in. generating circle and its flanks by a 3-in. generating circle. Determine the size of other generating circles that might have been used to generate the same tooth curves.

5. A 4-pitch gear is 12 in. in diameter. Determine the number of teeth and the circular pitch.

6. A gear 5 in. in diameter is to be driven by a pinion 3 in. in diameter. Both have cycloidal teeth with radial flanks. The addenda are to be $\frac{3}{8}$ in.

(a) Show accurately the path of contact.
(b) Show accurately the angle of approach on the driving pinion.
(c) Show accurately the angle of approach on the driven gear.
(d) Show the maximum pressure angle.
(e) Show the length of the acting surface of the flank of each gear.

7. Can a 12-tooth cycloidal gear and an 8-tooth cycloidal pinion be used to transmit motion properly between two shafts if both gears have radial flanks? Give complete reasons for your answer.

8. Determine the smallest pinion that can be made to work with a 16-tooth gear.

9. An interchangeable set of cycloidal gears has a diametral pitch of 2 and radial flanks on the 12-tooth gear. The addenda are such that two 8-tooth pinions working together give a path of contact 1.6 in. long. Is it possible for an 8-tooth gear to drive a 7-tooth gear of this set?

10. A 35-tooth gear 7 in. in diameter has standard machine-cut teeth.

(a) What is the diametral pitch? (b) What is the tooth thickness? (c) What is the addendum distance? (d) What is the outside diameter? (e) What is the root diameter? (f) What is the total depth of tooth?

11. A standard 20-tooth gear has teeth of $1\frac{1}{2}$-in. circular pitch. What is the diametral pitch, tooth thickness, standard addendum and standard dedendum?

12. What are the standard diameters of generating circles that would be used to generate the outlines of each of the following interchangeable gears?

(a) 28 tooth 2-pitch;
(b) 49 tooth 2-pitch;
(c) 100 tooth 12½-in. pitch diameter;
(d) 86 tooth 4-pitch;
(e) 4-pitch 20-in. pitch diameter;
(f) 50 tooth 1-in. pitch;
(g) 30 tooth annular 3-pitch.

13. A gear 6 in. pitch diameter has 12 teeth of standard height. The tooth face curves are generated with a 2½-in. generating circle and the flanks with a 3-in. generating circle. If this gear mates with a 10-in. gear, draw the path of contact and determine the maximum pressure angle.

14. A pair of cycloidal gears of 4-in. and 8-in. diameters have 16 and 32 teeth. If these are standard interchangeable gears determine the diameters of the generating circles for the faces and flanks of each gear.

15. A 12-tooth pinion 6 in. in diameter drives a 16-tooth gear. The generating circles are 6 in. in diameter.

(a) Determine the maximum arc of approach and the maximum arc of recess.
(b) Determine the maximum pressure angle.
(c) If these teeth were made of standard height what would be the arc of contact?

16. Standard cycloidal interchangeable gears have radial flanks on a gear of 12 teeth.

(a) What is the generating circle diameter for a 2-pitch gear?
(b) What is the circular pitch for a 2-pitch gear whose pitch circle is 30 in. diameter?
(c) Can a standard cycloidal gear be made with less than 12 teeth? Give reason for answer.

17. Two 3-pitch, standard interchangeable cycloidal gears have 10 and 12 teeth respectively.

(a) What are the diameters of the pitch circles?
(b) What are the diameters of the generating circles?
(c) If the addendum on each gear is 1.3183 times the circular pitch, and the 10-tooth pinion drives clockwise, show the path of contact.
(d) What is the length of the arc of contact?
(e) Will this pair of gears operate properly? Give reasons for answer.

18. An annular gear having radial flanks operates with a pinion having radial flanks. The annular has 30, 2-pitch, cycloidal teeth.

(a) What is the diameter of the largest pinion that can be used?

(b) If the annular rotates clockwise at 90 r.p.m. what will be the speed and direction of rotation of the pinion?

19. An interchangeable set of cycloidal gears of 1 pitch has a describing circle of 7½-in. diameter. What is the smallest annular gear that will operate with a pinion having radial flanks?

20. A 6-in. pinion of an interchangeable set of cycloidal gears has a generating circle of 2½-in. diameter, and a diametral pitch of 4. What is the smallest annular gear that can operate with this pinion? What is the largest annular gear that can operate with this pinion?

21. An annular gear having 60 standard cycloidal interchangeable gear teeth is 10 in. in diameter. What is the diameter of the largest pinion that will operate properly with this annular gear?

22. An annular gear has 50 teeth of standard interchangeable cycloidal system and rotates at 100 r.p.m. What is the slowest speed at which the driven gear can run and how many teeth will it have?

23. In a cycloidal gear and annular, the annular is to be stationary and the number of teeth on the spur gear is to be such that a point on its pitch circle moves in a straight line. The annular has 24 teeth of 4 diametral pitch. What is the diameter of the spur gear?

24. What is the largest pinion that will work with a 20-tooth 62-pitch annular gear of a standard interchangeable set?

25. A 2-pitch 16-tooth pinion of a standard interchangeable set drives a rack. Draw the path of contact and determine the arc of contact. What are the angles of approach and of recess for the pinion?

26. An annular gear having radial flanks operates with a pinion having radial flanks. The annular has 30, 2-pitch, cycloidal teeth.

(a) What is the diameter of the largest pinion that can be used?

(b) If the annular rotates clockwise at 90 r.p.m., what will be the speed and direction of rotation of the pinion?

27. Using Grant's Odontograph table, lay out the tooth curves for a pair of 6-in. and 4-in. standard cycloidal gears of 3 diametral pitch with the 4-in. gear driving.

When contact moves from the midpoint on the face of the driving gear to the pitch point, determine the amount of sliding between the tooth surfaces.

CHAPTER XV

INVOLUTE GEARS

210. The Involute Tooth Curve.* An involute of a circle is a curve the normals of which are all tangent to the same circle, called the ***base circle***. This involute can be generated by a tracing point on a non-elastic string as the string is unwound from the base circle. The involute can also be generated by a tracing point at the origin of a logarithmic spiral, which is rolled on the outside of the pitch circle, just as the cycloidal curves are generated by a tracing point on a describing circle. This method of construction, however, is clumsy, and the involute as generated from a base circle is used for convenience.

In Fig. 217, consider two circles with centers at A and B, with a string ef wrapped around each and having a tracing point t. If A is turned clockwise, the string will wind up on A, tracing on A, extended, the involute cta, and will unwind from B, tracing on B, extended, the involute btd. These involutes are traced simultaneously, and are always in contact at t; and the normal to both curves at the point of contact is always the line ef. The normal ef always intersects the center-line AB at the same point p. These involute curves can be used as the curves of gear teeth on A and B, and will give a constant angular velocity ratio, since we have

$$\frac{\omega_a}{\omega_b} = \frac{Bp}{Ap} = \text{a constant.}$$

The point p is then the pitch point, and Ap and Bp are the pitch-circle radii. Since the point of contact t is always on the line ef, the path of contact is a straight line, and the angle of obliquity or pressure angle θ is constant, differing in this respect from cycloidal tooth curves.

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* Advanced students and others particularly interested in toothed gearing are referred to the following publications: *Gearing*, by Geo. B. Grant, Philadelphia Gear Works. *Practical Treatise on Gearing*, Browne and Sharpe Manufacturing Co. *The Involute Gear*, *The Stub-tooth Gear*, Fellows Gear Shaper Co.

An *important property of the involute tooth curve is that the centers A and B can be separated,* as in Fig. 218, *without affecting the velocity ratio.* This may be proved as follows.

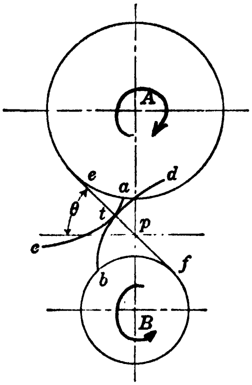

FIG. 217

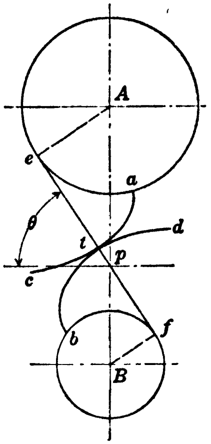

FIG. 218

Draw radii Ae and Bf. These lines will both be perpendicular to ef, and the triangles Aep and Bfp will be similar. Hence we have

$$\frac{\omega_a}{\omega_b} = \frac{Bp}{Ap} = \frac{Bf}{Ae} = \text{constant.}$$

In the location of the bearings supporting the gears, it is difficult to make the center distance an exact predetermined amount. With involute gear teeth, small errors in this center distance do not destroy the perfect action of the teeth. With cycloidal teeth, the case is different.

It is evident, however, that the angle of obliquity increases as the center distance is increased, and may become so great that the gears would be locked by friction. In practice, the angle of obliquity desired is decided upon, and the center distance made to correspond. Thus mating gears with pitch circles of radii Ap and Bp in Fig. 219 are to have an angle of obliquity of 14½°. A line bpa is drawn, making an angle θ equal to 14½°, with the tangent to the pitch circles at p. A circle of radius Aa, tangent to bpa, is the base circle from which the involute is generated.

Similarly, the base circle of the mating gear will have a radius *Bb*. These gears would be called 14½° involute gears.

211. Normal Pitch. In Fig. 219, it is evident that a line drawn normal to one tooth curve is also normal to the corresponding curve on the adjacent tooth, and that the normal distance between the tooth curves is constant and equal to the arc measured between the curves on the base circle. This distance is called the normal pitch, and is denoted by p_n. We have then

$$\frac{\text{normal pitch}}{\text{circular pitch}} = \frac{p_n}{p} = \frac{\text{circumference of base circle}}{\text{circumference of pitch circle}} = \frac{Bb}{Bp} = \cos\theta,$$

whence

$$p_n = p\cos\theta. \tag{94}$$

212. Limits of the Path of Contact. In § 210, it is shown that the path of contact is a straight line tangent to the two base circles, as *apb* in Fig. 219. Since the involute curve is generated by unwinding a non-elastic cord from the base circle, there is no curve below the base circle. Contact cannot begin beyond the point *a* and cannot continue beyond the point *b*, where the path of contact is tangent to the base circles. Any part of an involute tooth curve on the gear *A* extending beyond a circle of radius *Ab* is useless for driving the gear *B*. In fact, as shown in § 213,

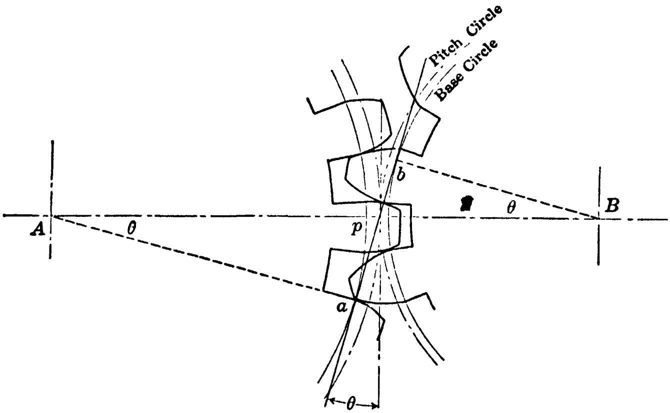

FIG. 219

it will cause interference. Similarly, any part of an involute tooth curve on the gear *B* extending beyond a circle of radius *Ba* is useless.

This is equivalent to saying that the maximum addendum radii for correct tooth action are *Ab* and *Ba* respectively, and that the maximum path of contact is *ab*. The maximum normal pitch will also be *ab*, and the maximum circular pitch will have the value $ab/\cos\theta$.

213. Interference in Involute Gears. It is shown in § 212 that contact on involute gears ends at the point of tangency of the base circle and the path of contact. There is no contact below the base circle, but the dedendum circle must pass below the base circle in order to provide clearance for the teeth of the mating gear. It is customary to make the flank radial between the base and dedendum circles, with a fillet joining this radial flank to the root circle.

In Fig. 220 are shown a pair of involute teeth in contact at *c*. The addendum of the gear *A* passes outside of the point *e*. Suppose the gear *A* to be rotated through the arc *df*. The tooth face will then be in the position *fg*. The gear *B* will rotate through the same arc, and its tooth curve will be in the position *bh*. The two tooth curves overlap as shown so that action is impossible unless the radial flank of *B* is cut back. This would make the flanks of *B* under-cut, a condition which is undesirable. Perfect action on the gears may be obtained if the addendum is cut back to a circle passing through the point *e*.

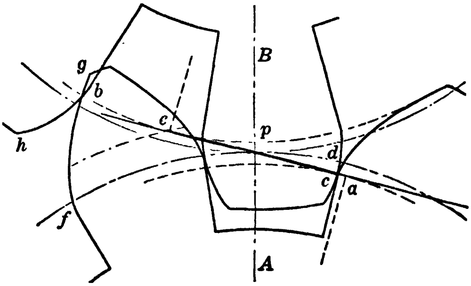

FIG. 220

If it is necessary to have the addendum extend beyond *e*, the top part of the face can be rounded off or made an epicycloid so that it will mate with the radial flank of the gear *B*, as is done in the proposed composite system.

214. Limiting Conditions for Involute Spur Gears. It has previously been pointed out that the addendum cannot extend beyond the point of tangency of the line of obliquity and the base circle of the mating gear unless it is modified from the true involute. Other limiting conditions are considered as special cases in the following discussions.

215. Minimum Angle of Obliquity. The least angle of obliquity that will permit continuous action with gears of known numbers of teeth can be found as follows.

In Fig. 219 the greatest possible normal pitch is *ab*, and the angle of obliquity is θ. Let T_a and T_b be the number of teeth on gear *A* and gear *B* respectively. Then we have

$$p_n = ab = (Ap + Bp) \sin \theta.$$

But

$$p_n = p \cos \theta, \qquad \text{and} \qquad (Ap + Bp) = \frac{(T_a + T_b)p}{2\pi}.$$

Hence

$$(95) \qquad p \cos \theta = \frac{(T_a + T_b)p \sin \theta}{2\pi}, \qquad \tan \theta = \frac{2\pi}{T_a + T_b}.$$

The angle θ is the smallest angle that will give continuous action with two gears having T_a and T_b teeth.

216. Smallest Equal Gears of Given Obliquity. With equal gears, T_a equals T_b, and the equation for the smallest angle of obliquity becomes

$$\tan \theta = \frac{\pi}{T}.$$

Thus for gears having an obliquity of 14½°, the least number of teeth with equal gears is

$$T = \frac{\pi}{\tan 14\frac{1}{2}^\circ} = \frac{\pi}{0.2586} = 12.11.$$

Theoretically, a pair of 12-tooth 14½° involute gears will not operate together, but the error is so small that in practice these gears would operate. It is better, however, to use pinions of not less than 15 teeth when possible.

217. Maximum Pitch. Given the pitch diameters and the obliquity, it is desired to find the maximum pitch.

From Fig. 219, the maximum normal pitch is

$$p_n = ab = (Ap + Bp) \sin \theta, \quad p_c = \frac{p_n}{\cos \theta},$$

$$\text{(96)} \quad \max p_c = (Ap + Bp) \tan \theta = \tfrac{1}{2}(D_a + D_b) \tan \theta.$$

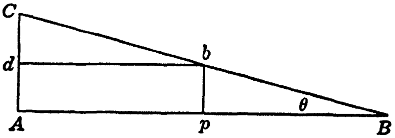

FIG. 221

The maximum pitch can be determined graphically. In Fig. 221, draw ApB equal to the sum of the given radii. Draw BC, making angle ABC equal to the given obliquity θ. Draw AC perpendicular to AB. Then AC is the required maximum pitch, since

$$\max p_c = (Ap + Bp) \tan \theta = AC.$$

In a comparison of Fig. 221 and Fig. 219, it also follows that pb drawn perpendicular to AB is equal to the arc of recess, and dC equal to AC minus pb is equal to the arc of approach.

218. Pointed Teeth. In every case the size of the gear may be limited by the faces of the teeth meeting at a point. Mathematically, this case is rather involved, and no exact solution is possible except by cut-and-try methods. Graphically, the gears may be checked as follows. In Fig. 222, the pitch circle, addendum circles, and path of contact have been found. Through the point b, draw the involute bc, the tooth curve of the gear A. Draw bA. If dc is less than ¼ of the circular pitch, the gears will not be pointed. If dc is greater than ¼ of the circular pitch, the teeth will be pointed below the addendum circle, and the desired path of contact cannot be obtained. The diameter of A must then be increased.

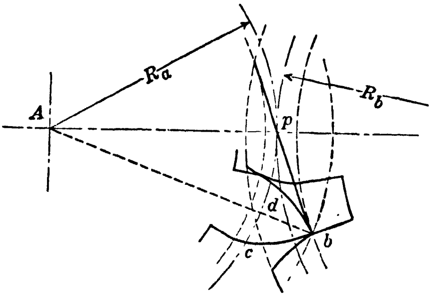

FIG. 222

219. Special Small Involute Pinions.* For the transmission of small amounts of power, as in watches and similar mechanisms, it is desirable to use pinions with small numbers of teeth. Theoretically, the smallest equal pair of involute radial-flank gears that will work have 4.625 teeth and an obliquity of 34° 12′, so that a 5-tooth gear is the smallest that will work with an equal gear.

With a rack and pinion, the pinion limit is 4.34 teeth with an obliquity of 38° 12′. Thus a five-tooth gear is the smallest that may be made to work with a rack. Although a four-tooth pinion will not mate with any spur-gear or rack, it will work with any internal or annular gear having from 6 to about 10,000 teeth, or practically a rack.

For an interchangeable set, that is, one in which the smallest gear will mate with any gear up to and including a rack, the theoretical pinion limit is 5.28 teeth with an obliquity of 35° 42′. Thus an interchangeable set can be made with 6 as the number of teeth on the smallest pinion.

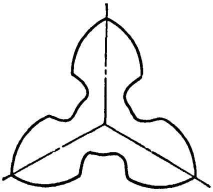

FIG. 223

If the flanks of the gears may be undercut, as in Fig. 223, a pair of 3- and 4-tooth pinions may be made to work together.

If the gears are to rotate in one direction only, pairs of one- and two-tooth pinions may be made to work together.

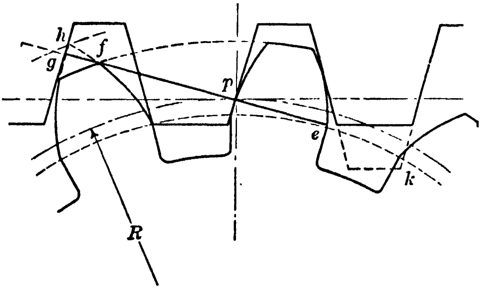

FIG. 224

220. Involute Rack. In Fig. 224 an involute pinion is shown driving a rack. The base circle of the rack is tangent to the line of obliquity *pe* at infinity and is, therefore, of infinite radius. The tooth curve of the rack being an involute generated on the base circle of infinite radius is a straight line perpendicular to the line of obliquity. Contact may begin at *e* and continue until the addendum of the pinion intersects the line of contact at *f*, the length of the line of contact being limited only by the teeth of the pinion becoming pointed at *h*. The limit of the path in this figure would be at *g*. If the rack tooth is extended beyond *e*, there will be interference as shown at *k*.

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* For a mathematical treatment of the limits of involute gearing, the reader is referred to the paper by A. B. Cox, "Limiting Cases in Involute Spur Gearing," published in Mechanical Engineering, Volume 46, No. 11, Nov., 1924.

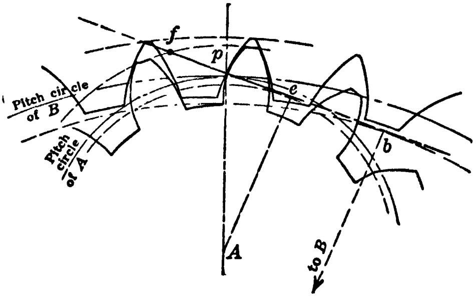


FIG. 225

221. Involute Annular Gears. In Fig. 225 an involute pinion is shown driving an involute annular gear. Contact may begin at *e* and, as in the case of the rack, will continue along *ep* until the addendum of the pinion cuts the line of contact at *f*. The base circle of the annular will be tangent to the line of contact at *b*, but contact cannot begin at this point, since interference will occur at all points between *e* and *b*.

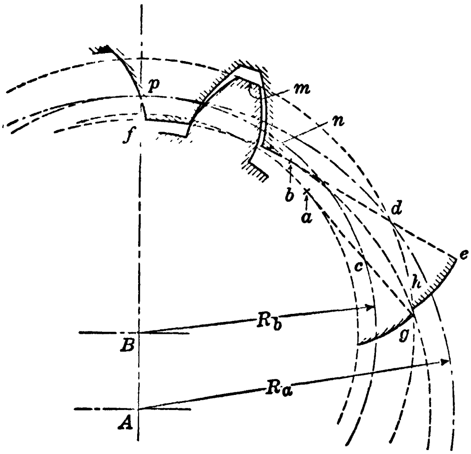


FIG. 226

In general, the arc of contact with annular gears is larger than with spur-gears of the same tooth height. They have less sliding, less wear, higher efficiency, and smoother action.

222. Interference in Involute Annular Gears. If the pinion approaches the annular in size, there may be interference caused by the point of the pinion tooth, *m*, striking the point of the annular tooth, *n*, in Fig. 226. To test for this interference, locate the point *g*, where the addendum circles intersect. Draw *gca* tangent to the pinion base circle, cutting the pinion pitch circle at *c*. Lay off the arc *pd* on the annular pitch circle, equal to the arc *pc*. Draw *bde* tangent to the annular base circle. Make *de* equal to *cg*. The point *e* is then a point on the annular tooth curve. Draw the involute *eh* on the annular gear. If *h* on the annular gear addendum falls between *f* and *g*, there will be interference. If *h* falls outside of the arc *fg*, there will be no interference.

The Fellows Gear Shaper Company in the pamphlet *The Internal Gear*, makes the statement that for the teeth to clear, *the pinion with standard 14½° involute teeth must have at least 12 teeth less than the annular gear*, and that with *20° stub teeth, the pinion must have at least 7 teeth less than the annular gear.*

223. Standard Involute Gearing. The generally accepted standard interchangeable gear has an obliquity of 15° for cast teeth, and 14½° for machine-cut teeth. The proportions of the teeth are the same as for cycloidal teeth as given in Table 3, page 243. With these proportions, 32 teeth in the 14½° system is the smallest number of teeth for a true involute gear which is interchangeable with all gears from an equal gear to a rack, since for smaller gears the addendum circle will pass beyond the interference point.

In § 216, it is shown that a pair of 14½° 12-tooth pinions will work together if the addendum circle passes through the interference points. With standard addenda, the addendum circle is always outside the interference point for gears with less than 32 teeth, and these gears must have their teeth rounded off beyond the interference point of a mating 12-tooth pinion.

224. Drawing the Involute Tooth Curve. The involute curves can be closely approximated by two circle arcs. Draw the pitch, addendum, dedendum, and base circles. For 14½° obliquity, the base circle may be taken 1/60 of the pitch diameter inside of the pitch circle. Find the face radius from Table 5, and, using a center on the base circle, draw in the face curve. Repeat for the flank between the pitch and base circles. Complete the flank with a radial line from the base circles to the dedendum circle. Connect the radial flank and the dedendum circle with a fillet of radius equal to 0.05 times the circular pitch.

TABLE 5

GRANT'S ODONTOGRAPH TABLE FOR 15° INVOLUTE GEARS

Teeth	Divide by the Diametral Pitch		Multiply by the Circular Pitch	
	Face Radius	Flank Radius	Face Radius	Flank Radius
10	2.28	.69	.73	.22
11	2.40	.83	.76	.27
12	2.51	.96	.80	.31
13	2.62	1.09	.83	.34
14	2.72	1.22	.86	.39
15	2.82	1.34	.90	.43
16	2.92	1.46	.93	.47
17	3.02	1.58	.96	.50
18	3.12	1.69	.99	.54
19	3.22	1.79	1.03	.57
20	3.32	1.89	1.06	.60
21	3.41	1.98	1.09	.63
22	3.49	2.06	1.11	.66
23	3.57	2.15	1.13	.69
24	3.64	2.24	1.16	.71
25	3.71	2.33	1.18	.74
26	3.78	2.42	1.20	.77
27	3.85	2.50	1.23	.80
28	3.92	2.59	1.25	.82
29	3.99	2.67	1.27	.85
30	4.06	2.76	1.29	.88
31	4.13	2.85	1.31	.91
32	4.20	2.93	1.34	.93
33	4.27	3.01	1.36	.96
34	4.33	3.09	1.38	.99
35	4.39	3.16	1.39	1.01
36	4.45	3.23	1.41	1.03
37–40	4.20		1.34	
41–45	4.63		1.48	
46–51	5.06		1.61	
52–60	5.74		1.83	
61–70	6.52		2.07	
71–90	7.72		2.46	
91–120	9.78		3.11	
121–180	13.38		4.26	
181–360	21.62		6.88	

For the ***rack teeth,*** draw straight lines at an angle of 75½° with the pitch line. Draw the outer half of the face with a radius equal to 0.67 times the circular pitch, and with a center on the pitch line.

For *internal gears,* draw the curves the same as for a spur gear. Make the addendum pass through the interference point. The pinion teeth do not require the standard dedendum, but may have the dedendum circle drawn in to clear the addendum circle of the annular by 0.05 times the circular pitch. Check for interference by the method in § 223.

225. The Stub-Tooth Gear. For many years, the 14½° involute gear has been the standard, although for special purposes some 20° and 22½° involute gears have been manufactured. It has been seen that with the standard addendum used on 14½° involute gears, it is necessary to round off the upper part of the tooth curve to prevent interference. Also, with 14½°, there is more sliding action between the tooth surfaces and consequently greater wear than with 20° involute teeth.

To overcome these faults, the stub tooth has been developed and is now extensively used, especially for automotive work. The most common and generally adopted stub-tooth system is the Fellows Stub Tooth. In this system, the 20° involute tooth curve is used, and the teeth are made shorter. Unfortunately, when the teeth were made shorter, the addendum and dedendum were not made proportional to the circular or diametral pitch, as in standard 14½° teeth. Instead, the Fellows stub tooth is designated by two diametral pitches as 2–3, 4–5, 6–8, etc., sometimes written 2/3, 4/5 and 6/8. The first figure is used to determine the pitch diameter, circular pitch, tooth thickness, and number of teeth. The second figure is used to determine the addendum and dedendum. Thus a 4–5 stub tooth has the same pitch diameter, circular pitch, and tooth thickness as any 4-pitch gear with the same number of teeth, while its addendum and dedendum are the same as a 5-pitch gear.

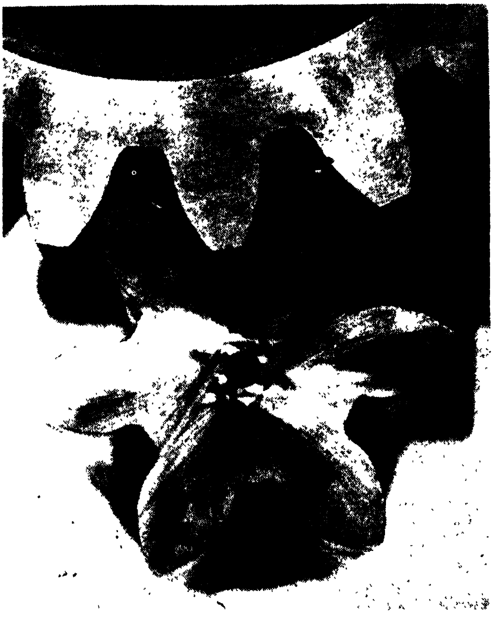
The 20° stub tooth is also stronger than the standard 14½° tooth, since it is shorter and thicker at the root circle. It has the disadvantage, however, that the actual normal tooth pressure and the separating pressure are slightly larger.

226. American Gear Manufacturers Association Stub Tooth. This standard is now in general use for the larger sizes of stub-tooth gears, and the Fellows Stub Tooth is in general use for the smaller sizes of stub-tooth gears.

In this system the tooth proportions are as follows.

Pressure angle	20°
Addendum $0.2546p$	$\frac{0.8}{P_d}$
Dedendum $0.3183p$	$\frac{1}{P_d}$

227. Long and Short Addendum Teeth. It is commonly known that tooth friction during the approaching action of gears is greater than during the recess action. It is therefore advisable to make all or nearly all of the action receding action. Since contact always begins between the flank of the driver and the face of the driven gear, this end can be accomplished by increasing the addendum of the driving gear and decreasing the adden-



Boston Gear Works, Inc., North Quincy, Mass.

FIG. 227. SIX-TOOTH INVOLUTE PINION.

dum of the driven gear. Figure 227 shows a six-tooth pinion of this kind.

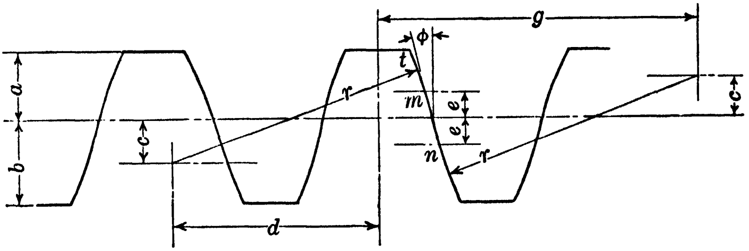
With involute gears, trouble may result if the pinion or smaller gear is the driven gear, since interference first occurs when the gear addendum extends beyond the interference point on the pinion. However, if the pinion is the driver, as is generally the case, these same conditions may prevent interference. When the pinion is the driver, we have the added advantage of stronger teeth, since the thickness of the tooth near the root circle is increased. In gear design, this is important, since the pinion tooth in gear sets having standard dimensions is always the weaker.

A special case, therefore, may be considered when the pinion is the driver, and when the number of teeth on the pinion is less than 32 for a 14½° system, 18 for a 20° system, or 14 for a 22½° system. The usual proportions of long and short addendum gear teeth for the above condition are as follows:

Pinion (Driver)	
Addendum	$\frac{1.4}{P_d}$
Dedendum	$\frac{0.7571}{P_d}$
Tooth thickness at pitch circle	
14½° obliquity	$\frac{1.778}{P_d}$
20° obliquity	$\frac{1.862}{P_d}$
Gear (Driven)	
Addendum	$\frac{0.6}{P_d}$
Dedendum	$\frac{1.5571}{P_d}$
Tooth thickness at pitch circle	
14½° obliquity	$\frac{1.364}{P_d}$
20° obliquity	$\frac{1.280}{P_d}$

228. Maag Gears. These gears are a Swiss development, the American rights being controlled by the Niles-Bemont-Pond Co. In each individual gear problem, the part of the true involute which will have the least sliding action and no interference, is chosen. The long and short addendum idea is used, but not the proportions given in § 227. The teeth are machine-generated by a rack cutter having straight sides. Because of the increased thickness of the pinion teeth at the root and the shorter length of teeth, they are stronger than standard teeth. Five-tooth pinions are possible, allowing high gear ratios, although ten-tooth pinions are recommended for general service.

229. Double Curve Teeth. In § 223, it is shown that most gears with true involute tooth curves will have interference if



P = Diametral pitch
$a = 1/P$
$b = 1.157/P$
$c = 0.56278/P$

$d = 2.94243/P$
$e = 0.37614/P$
$r = 3.750/P$

mn = Straight line (involute)
$\phi = 14\frac{1}{2}°$
mt = Cycloid

Fig. 228

the addenda are made standard, that is, $1/P_d$ in height. To prevent this interference, the tops of the teeth must be rounded off, or the flanks must be undercut to give the necessary clearance. If the outer part of the face and that part of the flank below the base circle of the involute curves are made of mating cycloidal curves, there will be no interference. Tentative standards for these gears have been approved by the American Standards Association. The proposed tooth outlines are as shown in Fig. 228. The outlines shown are for the basic rack of the system. The part of the tooth curve lying near the pitch line is straight and corresponds to a 14½ degree involute rack. The outer portion of the face and the inner portion of the flank correspond to the curves of a cycloidal rack. Any spur gears generated so that they will mate with this basic rack will of course operate correctly with each other.

TABLE 6

TOOTH PROPORTIONS FOR 14½ DEGREE COMPOSITE SYSTEM *

	In Terms of Diametral Pitch (Inches)	In Terms of Circular Pitch (Inches)
1. Addendum	$= \frac{1}{P_d}$	$= 0.3183p$
2. Minimum Dedendum	$= \frac{1.157}{P_d}$	$= 0.3683p$
3. Working Depth	$= \frac{2}{P_d}$	$= 0.6366p$
4. Minimum Total Depth	$= \frac{2.157}{P_d}$	$= 0.6866p$
5. Pitch Diameter	$= \frac{N}{P_d}$	$= 0.3183NP_d$
6. Outside Diameter	$= \frac{N+2}{P_d}$	$= 0.3183(N+2)p$
7. Basic Tooth Thickness on Pitch Line	$= \frac{1.5708}{P_d}$	$= 0.5p$
8. Minimum Clearance	$= \frac{0.157}{P_d}$	$= 0.05p$

PROBLEMS

1. Using a base circle of 2 in. diameter, lay out accurately to full scale a true involute curve for a tooth which is to extend 1¼ in. above the base circle.

2. The base circles of two involute gears are 1¾ and 2½ in. in diameter. Their center distance is 2½ in. What will be the pressure angle of the involute teeth generated from these base circles? What will be the pressure angle if the center distance is made 2¾ in.? State the pitch diameters in both cases.

3. Lay out a true involute tooth curve for a gear of 3 in. pitch diameter. The addendum distance is to be ½ in. and the pressure angle 15°.

4. A pair of involute gears having an angular velocity ratio of 2 to 5 have a circular pitch of 1.0472 in. and a normal pitch of 0.9675 in. If the shafts are 21 in. apart, determine the diameters of the pitch circles and of the base circles.

5. Let $P =$ diametral pitch, $p =$ circular pitch, $T =$ number of teeth, $R =$ diameter generating circle, $\theta =$ angle of obliquity, $D =$ pitch diameter,

---

* The tabulated data were taken from a pamphlet on *Spur-Gear Tooth Form* published by the American Society of Mechanical Engineers.

The term diametral pitch is used up to $P_d = 1$ inclusive, and the term circular pitch is used for $p = 3$ in. and over.

Given:	*Find:*
$T = 30,\ P = 4/5$	D
$P = 6/8,\ D = 4$	θ
$D = 100,\ p = 2$	T
$R = 3,\ D = 12$	T
$P = 6,\ \theta = 15$	R

6. Can two standard 3-pitch, involute spur gears of 21 teeth each be used to give constant velocity ratio to two shafts $7\frac{3}{8}$ in. apart? Give complete reasons for answer and prove by graphical construction.

7. An involute gear has 20 teeth of 2-diametral pitch. The pressure angle is 20°. What is the circular pitch and the normal pitch?

8. A pair of gears with 20 and 12 teeth have 3-diametral pitch. The pressure angle is $14\frac{1}{2}°$.
   (a) What is the circular pitch?
   (b) What is the normal pitch?
   (c) What is the maximum length of the path of contact?

9. Two gears having 8- and 12-in. pitch diameters with true 15-degree involute tooth curves are to operate together. What is the maximum pitch that can be used?

10. A 16-tooth 15-degree involute pinion of 2-diametral pitch operates with a rack. Determine the path of contact and the maximum tooth heights.

11. A 30-tooth true involute gear meshes with a 20-tooth gear. The diametral pitch is 4 and the pressure angle 20°.
   (a) Determine the circular pitch.
   (b) Determine the maximum length of the path of contact.
   (c) Determine the maximum arc of contact.
   (d) Determine the maximum addendum for each gear.

12. A 20-degree true involute gear of 3-pitch is 14 in. in diameter and meshes with a 12-in. gear.
   (a) What is the least pressure angle that can be used?
   (b) What will be the maximum addenda on these gears?

13. A pair of involute spur gears have base circles of 23.233 and 11.617 in. diameter, and the center distance is 18 in.
   (a) Determine the pitch diameters.
   (b) Determine the velocity ratio.
   (c) Determine the maximum length of the path of contact.
   (d) Determine the angle of obliquity.

14. What will be the number of teeth on the smallest pair of equal true involute gears that will operate with pressure angles of 15°, 20°, and $22\frac{1}{2}°$?

15. (a) Find the diameter and number of teeth of the smallest 3-pitch pinion with 20-degree involute teeth which will allow an arc of recess equal to the arc of approach equal to the circular pitch, and draw in the addendum and dedendum circles.
    (b) If the pinion drives a rack, what is the greatest allowable addendum for the rack?

16. Two gears of 20 and 40 teeth have true involute tooth curves. Determine the smallest pressure angle that can be used.

17. Will a true 14½-degree involute tooth gear of 12 teeth operate correctly with a rack with straight-sided teeth, using an addendum equal 0.3183p, and a dedendum equal 0.3683p? Give reasons for your answer.

18. Using true 14½-degree involute tooth curves and addenda equal to 0.3183p and dedenda equal to 0.3683p, determine whether a 16-tooth and a 32-tooth gear can operate together. Give complete reasons for your answer. Can these gears be made to work? Give proof of your answer.

19. (a) Prove that two given involute gears can have their center distance changed without destroying the correct tooth action.
    (b) Is it possible to have an involute gear operate correctly with a 20-degree involute gear and a 22½-degree involute gear at the same time?

20. (a) Determine graphically whether a pair of 10-tooth true involute gears can be made to work together, using a 14½-degree obliquity.
    (b) If the addenda of these gears were standard height would they work together?

21. Sketch two involute gears of unequal diameters with the smaller gear driving clockwise. Assume the angle of obliquity and the addendum circles such that there will be no angle of approach, but a maximum angle of recess.

22. Lay out a rack and pinion. Number of teeth in pinion, 18; pitch, ¾ in.; addenda, 0.3p; height of teeth, 0.7p; diameter of shaft, 1 in.; thickness of rack, ⅝ in.; faces of pinion and flanks of rack teeth involute. Faces of rack and flanks of pinion teeth cycloidal; generating circle half the diameter of the pitch circle.

23. Sketch a 15-degree involute pinion driving simultaneously with a 15-degree rack and a 22½-degree rack, indicating the path of contact, pitch diameter, and addendum in each case. Make the rack addendum as large as possible without having interference, and give its numerical value.

24. (a) What is a stub-tooth gear?
   (b) A 4–5-pitch gear has 20 teeth. Determine the pitch diameter, circular pitch, and outside diameter.

25. A 6–8 Fellows stub-tooth gear has 20 teeth.
   (a) What is the pressure angle?
   (b) What is the circular pitch?
   (c) What is the addendum distance?
   (d) What is the outside diameter?

26. Using Grant's Odontograph table, draw two complete teeth on each gear. Diametral pitch 1½. Standard addenda.
   (a) A 12- and an 18-tooth gear in mesh.
   (b) An 18-tooth pinion and a rack in mesh.

27. Using Grant's Odontograph table, draw two complete teeth on each gear. Circular pitch 1¼ in. Standard addenda.
   (a) A 16-tooth pinion in mesh with a 20-tooth gear.
   (b) A 16-tooth pinion in mesh with a rack.

CHAPTER XVI

TWISTED AND BEVEL GEARS

230. Reducing Gear Noise. In plain spur gears, the elements of the tooth surfaces are straight lines parallel to the axis of rotation. Contact between two tooth surfaces is theoretically line contact along these elements from the time contact begins until contact ceases. Contact begins across the entire width of the gear at one time, and at high speeds and heavy loads is conducive to noisy action, causing the gear to sing. Gears made of other materials than metal, such as rawhide, cloth, pressed paper, and wood, have been used to reduce this noise. In general, however, these gears are not so strong as iron and steel gears.

One of the best ways to reduce the noise, or singing, and to produce smooth-tooth action is to use some form of ***twisted gear,*** that is, a gear in which the elements of the tooth curves are not parallel to the axis of rotation.

231. Step Gears. These are sometimes known as ***Hooke's Gears.*** In Fig. 229, suppose the gear to be cut into slices, or to be made up of four narrower gears A, B, C, and D. In Fig. 230, B has been rotated forward an amount equal to ¼ of the circular pitch, C, ½ of the circular pitch, and D, ¾ of the circular pitch. In the figure, the tooth on gear A is just beginning contact at a. It is evident that on the gear B the tooth has already traversed a portion of the arc of contact, and contact is at b. Similarly, contact is at c and d on the gears C and D. Contact no longer begins across the entire width of the gear at any one time, but is divided. The shock of beginning contact is therefore divided, and the noise and impact lessened. Figure 231 shows two large step gears ready for service.

232. Helical Teeth. When the step gear is made up of a number of infinitely thin steps, we have continuous tooth surfaces whose elements are not parallel to the axis of rotation.

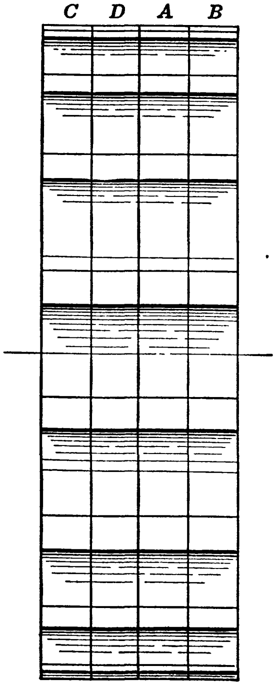
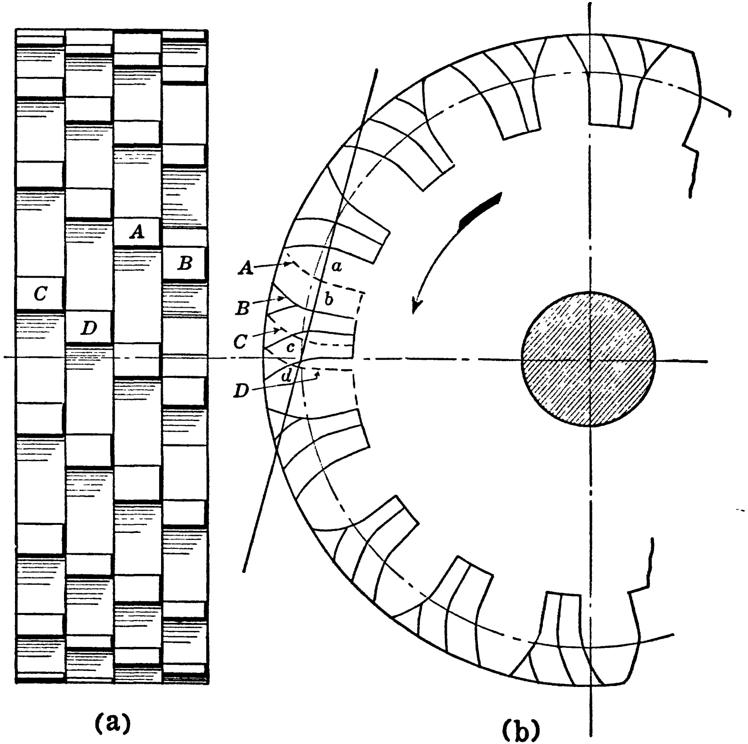
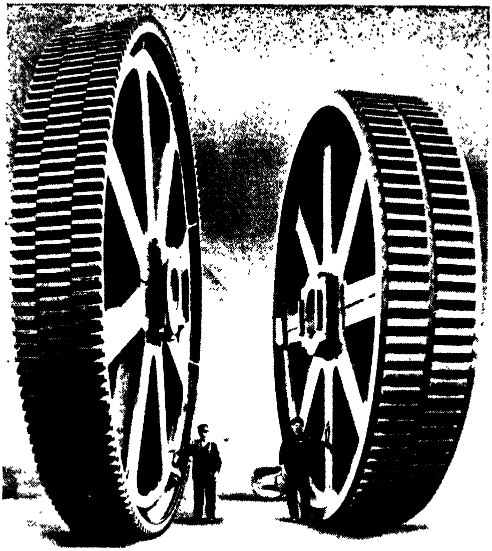


FIG. 229



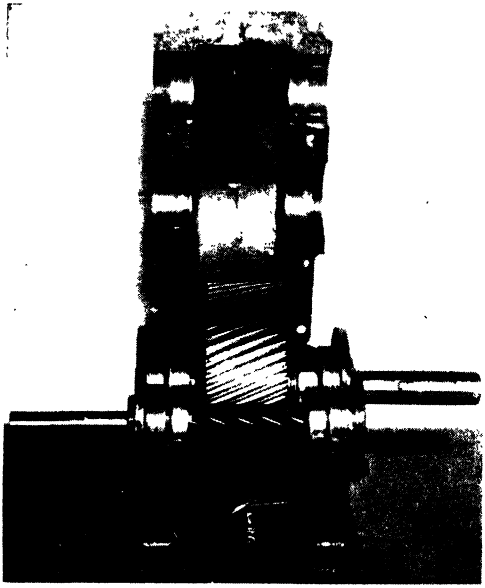
(a) (b)

FIG. 230



Mesta Machine Co., Pittsburgh, Pa.

FIG. 231. STEPPED GEARS.



Boston Gear Works, Inc., North Quincy, Mass.

FIG. 232. HELICAL GEAR AND PINION SPEED-REDUCER.

The amount of the twist in the gear teeth is usually uniform, in which case we have helical gears, the elements of the tooth surfaces being helixes. Note that with two mating gears, the helixes are opposite in direction, one being right-hand and one left-hand, Fig. 232.

Contact on any one tooth now begins at a single point, and as the gear revolves, this point contact becomes a line contact which lengthens until the maximum is reached, and then shortens until contact ceases. It is seen that some point on the tooth may be in contact at the pitch point, while other points are in contact with approaching action, and still others are in contact with receding action. This form of contact gives very smooth and noiseless action and makes this type of gear suitable for much higher speeds than is possible with plain spur gearing.

Helix angle. The pitch line, or intersection of the tooth surface and the pitch surface, is a helix, and the angle which this helix makes with the plane containing the axis of rotation is called the helix angle.

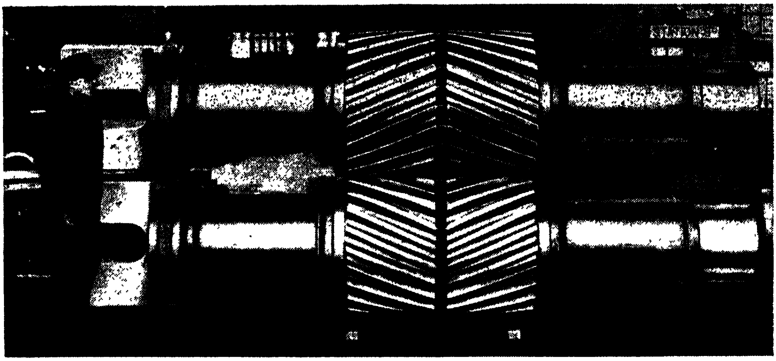
Normal section is the section of the tooth made on a helical surface perpendicular to the tooth.

Real pitch, or ***pitch,*** is measured on a plane perpendicular to the axis of rotation and corresponds to the pitch of a plain spur gear.

Normal pitch P_n is the pitch measured on the normal section and is equal to the circular pitch multiplied by the cosine of the helix angle. The normal diametral pitch P_n is the diametral pitch divided by the cosine of the helix angle. We may then write

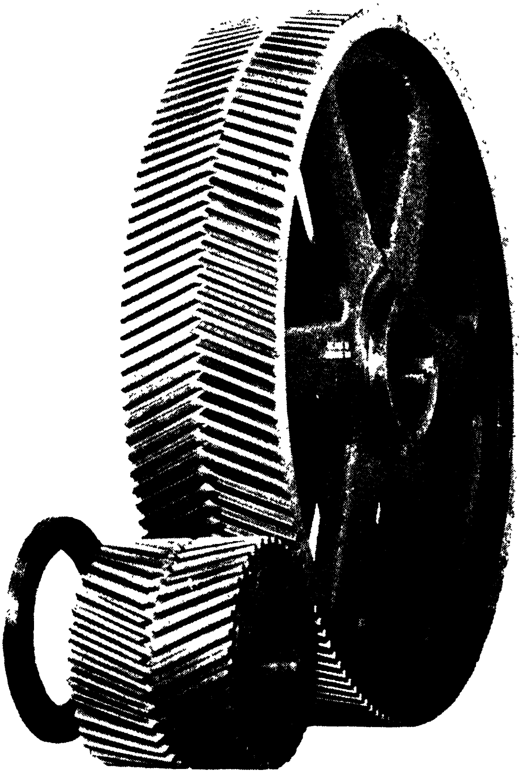
$$(97) \qquad P_n = \frac{P_d}{\cos\alpha}, \qquad \text{and} \qquad p_n = p\cos\alpha.$$

Since the shape of the tooth is determined by the normal pitch in the ordinary methods of cutting helical teeth, the cutters used should be selected on the normal pitch basis and not on the real pitch basis. For example, a 10-inch 30° helical gear has 20 teeth. The diametral pitch is 2, but the normal diametral pitch is 2/0.866 or 2.31. The cutter used should be selected for the



Mesta Machine Co., Pittsburgh, Pa.

FIG. 233. HERRINGBONE GEARS FOR A ROLLING-MILL DRIVE.



The Falk Corporation, Milwaukee, Wis.

FIG. 234. STAGGERED TOOTH HERRINGBONE GEARS.

number of teeth on the gear divided by the cube of the cosine of the helix angle, that is, for $20/\cos^3 30$ or 30.8, say 31 teeth, 2.31-pitch. The cutter used would probably be a $1\frac{3}{8}$-inch circular pitch 31-tooth cutter unless very accurate work justifies the expense of a special cutter.

End thrust. The force tangent to the pitch surface required to drive a helical gear is the same as for a spur gear, but the pressure between the teeth is normal to their surfaces. This normal force in the plane tangent to the pitch surface equals the tangential driving force divided by the cosine of the helix angle. This oblique action causes an end-thrust on the gear equal to the tangential driving force times the tangent of the helix angle, and this end-thrust must be taken on a thrust bearing, with an accompanying power loss.

233. Herringbone Gears. If two helical gears with the teeth twisted in opposite directions, but with the same helix angles, are mounted on the same shaft, the end thrust is eliminated. If the two gears are combined into one gear as in Figs. 233 and 234, we have a herringbone gear, so called on account of its likeness in appearance to the back-bone of a herring.

234. Minimum Width of Helical Gears. In order that there may be contact across the entire width of the gear, the width of the gear must be equal to, or greater than, the circular pitch divided by the tangent of the helix angle. We may then write

$$(98) \qquad f \geqq \frac{p}{\tan\alpha}, \qquad f \geqq \frac{\pi}{P_d \tan\alpha}.$$

With herringbone gears, the width will be twice the values given, plus the width of the center groove left for cutting tool clearance. Correct action can be obtained with gears of less width, but the pressure between the teeth will be concentrated on less than the full width of tooth.

235. Standard Herringbone Gears. In an effort to reduce the large variety of helical and herringbone gears being manufactured, the following standards have been adopted by the American Gear Manufacturers Association.

TABLE 7

Tooth Proportions for Herringbone Gears

Kind of teeth	involute
Pressure angle	20° measured in the plane normal to the axis of rotation
Helical angle	23° with the axis of rotation
Addendum	$\frac{0.8}{P_d}$ where P_d is the diametral pitch in the plane normal to the axis of rotation
Dedendum	$\frac{1}{P_d}$
Clearance	$\frac{0.2}{P_d}$
Whole depth	$\frac{1.8}{P_d}$
Minimum width of active face	$\frac{1.6}{P_d}$

For high-speed work, as in speed reducers for turbine-driven machinery, pitch-line velocities as high as 7000 feet per minute are common (12,000 feet per minute has been successfully used), and the helix angle is increased to 30 and 45 degrees.

236. Spiral Gears. When two shafts are not parallel and do not intersect, motion can be transmitted from one to the other by means of two helical gears having the same or different helix angles. Such a pair of gears is called spiral gears. The *angular velocities of such gears are not inversely proportional to the pitch diameters,* but have the following ratio

$$\frac{N_a}{N_b} = \frac{\omega_a}{\omega_b} = \frac{D_b \cos \alpha_b}{D_a \cos \alpha_a} \tag{99}$$

where

N = revolutions per minute,

ω = angular velocity,

D = pitch diameter,

α = helix angle of tooth.

The angle between the shafts and the helix angles must have the relation $\theta = \alpha_a + \alpha_b$, where θ is the angle between the axes of rotation as in Fig. 235, and α_a and α_b are right- and left-handed.

The real circular pitches measured on planes perpendicular to the axes of the gears are not equal, but the normal pitches must

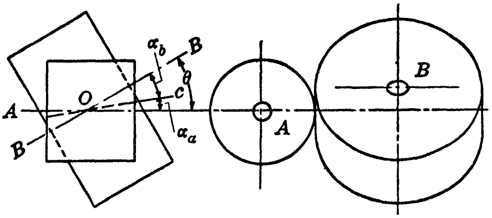


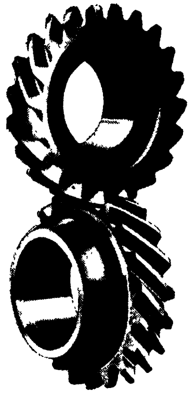
FIG. 235

be equal. The circular pitches have the ratio

$$(100)\qquad \frac{p_a}{p_b} = \frac{\cos \alpha_b}{\cos \alpha_a}.$$

The most common use of spiral gears is to transmit power between two shafts at right angles. If the two gears are exactly alike and are set at right angles, the velocity ratio will be 1 to 1. This combination is the usual combination referred to as spiral gears, Fig. 236.

Contact between the teeth of these gears is point contact only, instead of line contact as in helical gears with parallel shafts. For this reason these gears are not desirable for high power transmission. For line contact, the pitch surfaces must be hyperboloids as discussed in § 183.



W. A. Jones Foundry and Machine Co., Chicago, Ill.

FIG. 236. SPIRAL GEARS.

237. Bevel Gears. The gears previously considered were for use in transmitting motion and power between parallel axes or between axes which if not parallel did not intersect. If the axes meet at some point, we have a special case called ***bevel gearing.*** The rolling surfaces upon which the gear teeth are built are conical surfaces having a common apex instead of cylindrical surfaces as in spur gearing. Definitions for certain terms which are not used with spur gears, or which have slightly different meanings when applied to bevel gears, are given here. These are graphically represented in Fig. 237.

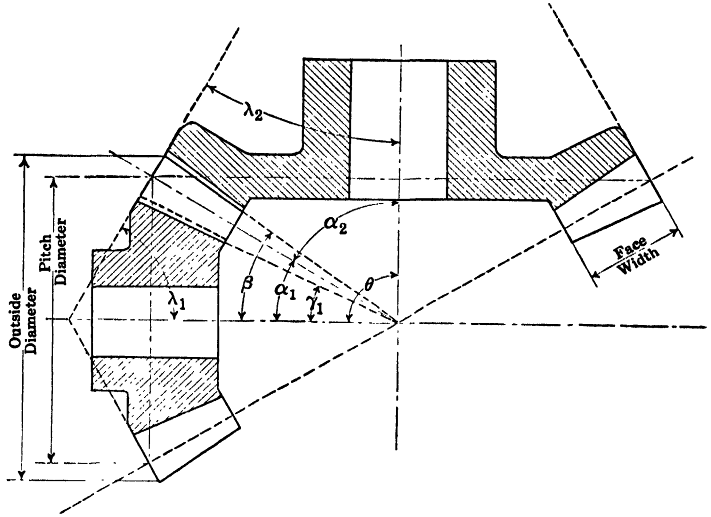


FIG. 237

Pitch cones are the imaginary rolling cones upon which the teeth are built.

Pitch circle is the circle forming the large end of the pitch cone.

Pitch diameter is the diameter of the pitch circle at the base or large end of the cone.

Pitch point is the point of tangency of the pitch cones at the large end.

Width of face is measured on an element of the pitch cone.

Shaft angle is the angle θ, between the two intersecting axes of rotation. Unless otherwise stated, this angle is understood to be 90°. When the shafts do not intersect at 90°, the angle θ is the one which includes the pitch point.

Center, apex, or ***cone angle*** is the angle α which an element of the pitch cone makes with the axis of rotation.

Face angle is the angle β between the top surface of the tooth and the axis of rotation.

Cutting angle is the angle γ between the surface forming the bottom of the tooth space and the axis of rotation.

Backing is the distance from the pitch circle to the outer end of the hub measured parallel to the axis of rotation.

Back cone is an imaginary cone whose base circle is the pitch circle of the gear, and whose elements are perpendicular to the pitch cone. The length of the elements of the back cone is called the back-cone radius, or formative radius.

Formative number of teeth is the number of teeth, of the same pitch as used on the gear, which would be found on a complete spur gear of radius equal to the back-cone radius.

238. Theory of Bevel Gearing. Bevel gears must be studied by reference to tooth outlines generated on spherical surfaces, the centers of which are the apexes of the rolling conical surfaces. After a few fundamental principles have been developed, the theory is the same as that previously developed for spur gearing.

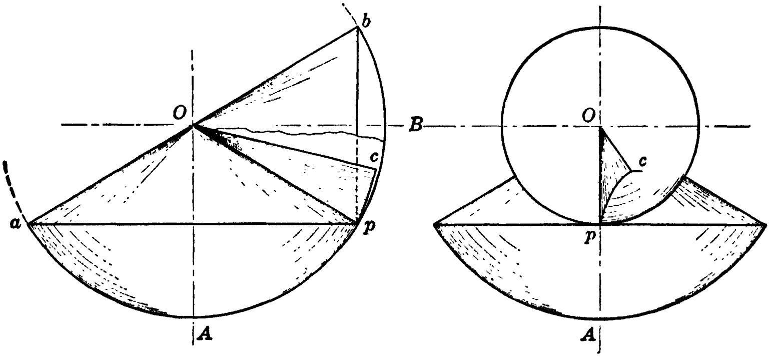


FIG. 238

In Fig. 238, *OA* and *OB* are the axes of two gears having conical pitch surfaces with the apexes at *O* and in contact along the pitch line *Op*. Let the point *p* be used as the tracing point of a templet which rolls on the pitch circle *ap*. The point *p* will generate a tooth curve *pc* on the surface of a sphere having a radius *Op*. The line *Op* will generate a tooth surface *Opc* for the gear whose pitch cone is *pOa*. The elements of this tooth surface are radii of the sphere. The same templet rolling on the pitch circle *bp* can be used to generate a tooth surface on the pitch cone *pOb*, which will mate with the tooth surface *Opc*.

The basic tooth curves commonly used for bevel gears are the same as for spur gears, namely, cycloidal and involute.

239. The Spherical Cycloidal Tooth. This tooth is generated by an element of a describing cone rolling on the pitch cone surface. In Fig. 239, *Op* is the element of a describing cone *G* which, when *G* is rolled on the outside of the cone *A*, will generate a tooth surface the intersection of which with the spherical surface having the radius *Op* will be the face tooth curve *pa*. When *G* is rolled inside of *B*, the point *p* will generate the flank curve *pb*. If these curves are generated simultaneously, the axis of the describing cone being held stationary and the pitch cones revolving together, it is readily seen that all tooth-curve requirements, as developed for spur gears, have been fulfilled.

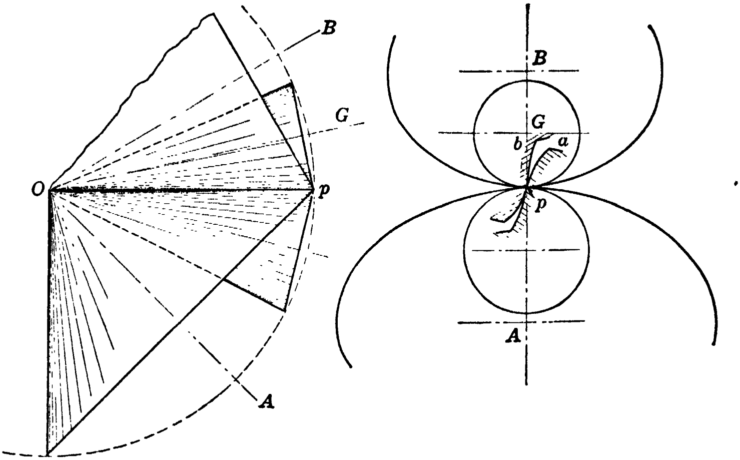


FIG. 239

The path of contact at the large end will be on a circle formed by the intersection of the cone *G* and the spherical surface. If the flank curves of *A* and the face curves of *B* are formed in a similar manner, the complete path of contact will be on two circles, which, however, are not in the same plane, as they are in spur gearing. The intersection of the pitch surfaces and the spherical surface are, of course, circles on planes normal to the axes of rotation. These circles are called the pitch circles.

240. The Spherical Involute Tooth. This tooth has for its path of contact a great circle of the sphere, which intersects the pitch circle at the pitch point. The tooth curve is generated when a great circle of the sphere is rolled on the base circle, which is the circle on the sphere parallel to the pitch circle and tangent to the path of contact.

241. Tredgold's Approximation. The tooth profiles at the large end of the teeth should be laid out on a spherical surface; but since the spherical surface cannot be developed into a plane surface, it is impossible to do so, in practice. The profile of the tooth curves on a conical surface tangent to the spherical surface at the pitch circle differs very little from the profile on the spherical surface.

These conical surfaces are called the ***back cones*** or ***normal*** cones. If these conical surfaces are developed on a plane surface, they form portions of circles which are then treated as ordinary spur gears with radii equal to the length of the elements of the back cone. The tooth curves for the large end of the bevel gears are laid out on these equivalent spur gears, called ***formative gears,*** and are then transferred to the bevel gear.

In Fig. 240 is shown a pair of bevel gears with the tooth outlines drawn by Tredgold's method. OA and OB are the gear axes, and Op is an element of each pitch cone. The back cone radii are ap and bp. Draw pe perpendicular to ap, and def parallel to ap. With a radius de equal to ap, and a radius fe equal to bp, draw the pitch circles of the formative gears, and construct the tooth curves in the same manner as for any pair of spur gears with radii de and fe. The drawing of the gears proper is a matter of orthographic projections. For illustration, the projections for one point on each gear tooth are given. Select a point g on the tooth of the formative gear of the gear A. With dg as a radius, draw a circle arc to intersect def at i. Draw ij perpendicular to apb. Through j pass a plane perpendicular to the axis OA, cutting the back cone in the line jk. Project j to l, and draw arc lm with sl as a radius, this circle being the intersection of the plane jk and the back cone. On the pitch circle *tu*, step off arcs equal to the circular pitch, and draw the center-lines of the gear teeth, as *Sn*, *Sr*, etc. On the arc *lm*, lay off on

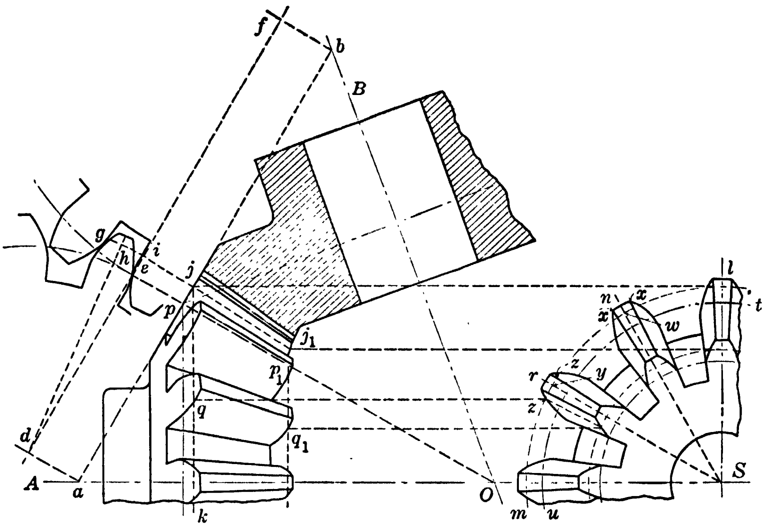
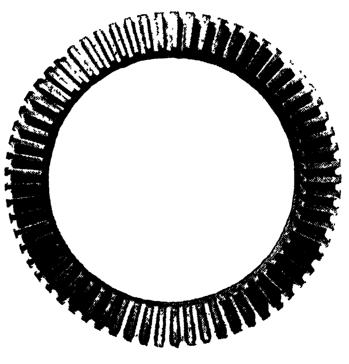


FIG. 240

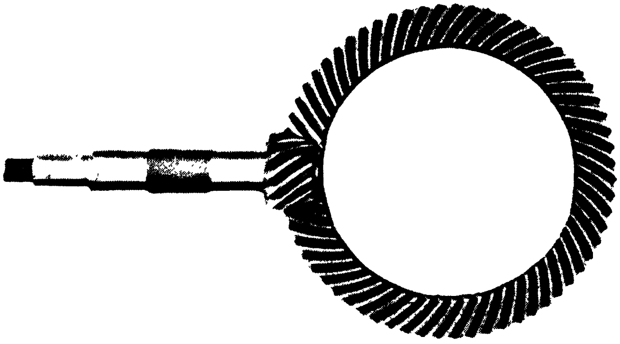
each side of these center lines distances *wx*, *yz*, etc., equal to *hg*. The points *x*, *z*, etc., are on the tooth curves. Project these points back to the line *jk*. Similarly, as many points as are desired can be found, and the tooth curves drawn through them. The small ends of the gear teeth are found in the same way.



Gleason Works, Rochester, N. Y.
FIG. 241. BEVEL GEAR.

242. Velocity Ratio of Bevel Gears. In § 178, the velocity ratios and center and apex angles of friction cones were discussed, and the relations developed in that article apply to bevel gears, since the pitch cones of the gears may be substituted for the rolling cones.

243. Spiral Bevel Gears. By twisting the teeth of ordinary spur gears, we obtained spiral or helical gears. Similarly, by twisting the teeth of bevel gears, as in Fig. 242, we obtain spiral bevel gears. These possess the same advantages as to noise and smoothness of action that we find in helical spur gears; also more end adjustment is possible than with straight bevel gears without noise or bearing trouble. About 30 percent higher velocity ratios can be used with spiral than with straight bevel gears.



Gleason Works, Rochester, N. Y.

FIG. 242. BEVEL GEAR AND PINION.

The curve of the teeth may be a spiral with the apex of the pitch cone as a center, but with most types of bevel gear generators used, the curve is a circular arc.

Angle of spiral is the angle which the tangent to the tooth at the center of the gear face makes with the element of the pitch cone.

Arc of the spiral or ***lead of the spiral*** is the distance that the spiral has advanced from the outer end of the tooth face to the inner end, measured between pitch-cone elements at the large end of the pitch cone. This arc should not be less than 1¼ times, and preferably not more than 1½ times the circular pitch.

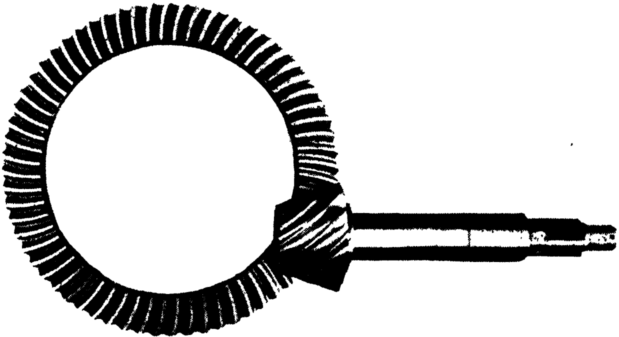
Right- and left-hand spirals. The angle of the spiral is designated by the spiral on the pinion, but it should be noted that the spiral of the gear is right-handed when the spiral of the pinion is left-handed.

To avoid undercut flanks, unequal addendum teeth are used, the present tendency being to alter the addendum height with the gear ratio, decreasing the addendum of the gear as the velocity ratio increases.

244. Skew Bevel Gears. In § 183, it was shown that when the axes did not intersect and were not parallel, a pair of hyperboloids of revolution would give perfect rolling contact. If the hyperboloids are used as pitch surfaces, teeth can be constructed on them, and we have ***hyperbolic gears.*** In addition to the rolling action, the hyperboloids slide together along the straight line of contact. This sliding makes the tooth construction difficult and also creates excessive friction.

If only a portion of these hyperboloids are used, we have ***skew bevel gears.*** The gear teeth, although having straight surface elements, are difficult to lay out and machine, and are very little used. No exact method of developing the tooth curves has been developed. For full discussion of this type of gear, the reader is referred to *Gearing* by G. B. Grant.

245. Hypoid Gears.* These gears were introduced in 1925 by the Gleason Works. They differ in appearance from spiral bevel gears chiefly in that the axis of the pinion does not intersect the axis of the gear, which in addition to producing smooth action permits the shafts to pass each other. These are essentially



Gleason Works, Rochester, N. Y.

FIG. 243. HYPOID GEAR AND PINION.

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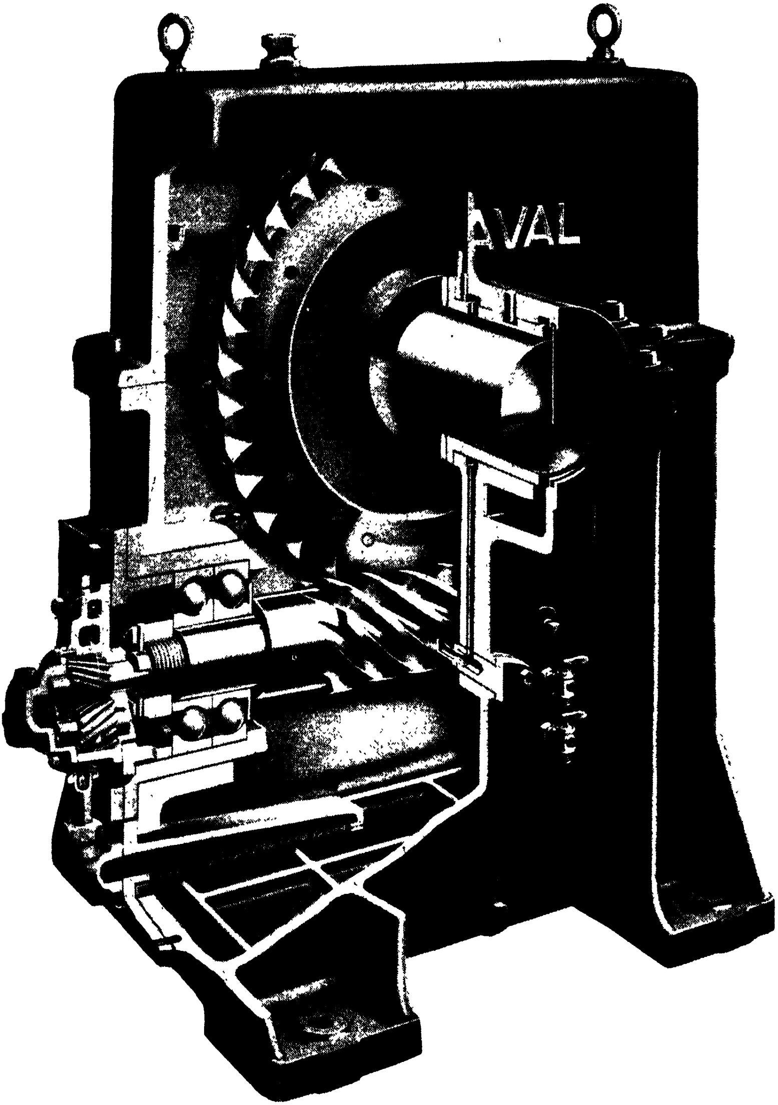
* See "The Design and Manufacture of Hypoid Gears," by A. L. Stewart and E. Wildhaber. Journal of the Society of Automotive Engineers, vol. 18, June, 1926. Also "Large Spiral and Hypoid Gears," by A. H. Candee. Transactions of the A. S. M. E., vol. 51, 1929.

hyperboloid gears with approximately conical pitch surfaces, with teeth generated by methods similar to those used for spiral bevel gears. Figure 243 illustrates a gear and pinion of this type.



DeLaval Steam Turbine Co., Trenton, N. J.

FIG. 244. WORM-GEAR SPEED-REDUCER.

246. Worm and Worm-Gear. Two shafts at right angles but not intersecting may have motion transmitted between them by a ***worm and worm-gear,*** or ***worm-wheel.*** This is a special case of helical gearing. The worm is formed by a continuous helical tooth similar to the thread on a bolt. There may be one, two, or more teeth forming single-thread or multiple-thread worms. Figure 244 illustrates a worm-gear drive.

The *velocity ratio depends on the number of threads on the worm and the number of teeth on the gear, and does not depend on the diameters and center distances.* Thus a double-thread worm working with a worm-gear of 40 teeth gives a velocity ratio of

$$\frac{\omega_w}{\omega_g} = \frac{40}{2} = 20,$$

or the worm makes 20 revolutions while the worm-wheel makes one revolution.

The teeth on the worm are of the involute system in order that the tooth curves on a plane through the worm axis will have straight sides like an involute rack, and the action of the worm and worm-gear is similar to that of an involute rack and pinion. The angle between the tooth surfaces on single-thread worms is usually 29 degrees, making them 14½-degree involutes. On multiple-thread worms, the angle is changed to meet design requirements of high efficiency, strength, etc. The worm pitch is usually made in circular pitch in order to enable the threads to be cut in an ordinary screw turning lathe.

The worm-gear teeth are cut by means of a hob having a form like the worm and fed radially into the revolving gear. This gives the pitch surface of the worm-gear a hollowed form which fits around the pitch surface of the worm.

PROBLEMS

1. A 6-in. helical gear has 30 teeth and a helix angle of 30°. Determine the real circumferential pitch and the normal pitch.

2. A 2-pitch helical gear is 10 in. in diameter and has a helix angle of 20°. Determine the real circumferential pitch and the normal pitch.

3. A 20-tooth helical gear has a normal pitch of ¾ in. and a helix angle of 30°. Determine the pitch diameter of the gear.

4. A 3-pitch helical gear 5 in. in diameter has a helix angle of 23°. For what pitch should the cutter be selected?

5. A pair of helical gears are used to transmit power from a shaft rotating at 150 r.p.m. to a shaft rotating at 350 r.p.m. The center distance is 5 in. and the helix angle of the gears is 23°. The smaller gear has 21 teeth.

   (a) What is the diametral pitch?
   (b) What is the circular pitch?
   (c) What is the normal pitch?
   (d) How wide must the gears be made so that there will always be contact at the pitch point?

6. A helical gear having 20 teeth of 2 real diametral pitch has a helix angle of 30°, and a tangential pressure of 85 lbs. at the pitch line. Determine the normal circular pitch, the end thrust, and the cutter to be used in cutting the teeth.

7. A herringbone gear of 24 teeth, 3 diametral pitch has a ¼-in. clearance groove at the center. Determine the minimum width of gear for a helix angle of 23°, and for a helix angle of 30°.

8. A pair of herringbone gears made according to the accepted standards has 30 teeth of 1½ diametral pitch. Determine the pitch diameter, helix angle, outside diameter, root diameter, and minimum face width.

9. A pair of equal diameter spiral gears are mounted on shafts 2 in. apart, with an angle between the shafts of 60°. If the velocity ratio is to be 2 to 1, find the helix angle and the circular pitch of each gear.

10. Two spiral gears are 4 and 5 in. in diameter and are to rotate at 100 and 150 r.p.m. respectively. The helix angle on the smaller gear is 15°. What is the helix angle on the larger gear, and what is the angle between the shafts?

11. Two bevel gears have a 3 to 1 velocity ratio. The larger has 48 teeth and a pitch diameter of 12 in. The shaft angle is 90°.

   (a) How many teeth has the small gear?
   (b) What is the pitch diameter of the small gear?
   (c) What is the formative gear radius for each bevel gear?
   (d) What is the formative number of teeth for each bevel gear?
   (e) What is the circular pitch of the formative gears?
   (f) What is the circular pitch of the bevel gears?

12. A pair of shafts intersecting at an angle of 120° are to have two to one velocity ratio. The pinion is to be 6 in. in diameter and of 2½ diametral pitch.

   (a) What is the diameter of the gear?
   (b) How many teeth are on the gear?

(c) If you were to lay out the tooth outlines or select cutters for these gears, how many teeth would you use in looking up the tooth outline radii or the cutters to be used?

13. A pair of shafts intersecting at an angle of 75° rotate at 80 and 320 r.p.m. The smaller gear is the driver and has 16 teeth, 4 diametral pitch. Determine the following items:

(a) Center or pitch angle of each gear.
(b) Total height of tooth.
(c) Outside diameter of gears.
(d) Formative number of teeth on each gear.

14. A pair of bevel gears with a velocity ratio of 2.5 to 1 have their axes at right angles. The diametral pitch is 5 and the larger gear has 25 teeth.

(a) What is the pitch diameter of the large gear?
(b) What is the pitch diameter of the small gear?
(c) What is the pitch angle of the large gear?
(d) What is the outside diameter of the large gear?
(e) What is the back-cone radius of the large gear?

15. Show graphically how to determine pitch diameter and outside diameter of blanks for a bevel pinion and crown gear. Angle between shafts is 120°. Pitch diameter of pinion is 6 in. What will be the number of teeth and exact outside diameters for each of these gears? In determining the tooth outline, how many teeth would you consider to be on each of these gears?

16. A triple-thread worm drives a worm-gear of 72 teeth. The circular pitch is $1\frac{1}{2}$ in., and the pitch diameter of the worm is 4 in.

(a) What is the helix angle?
(b) What is the velocity ratio?
(c) What is the diameter of the worm-gear?
(d) Could the worm be changed so that the same worm-gear may be used to transmit a 26 to 1 velocity ratio? Give reasons for your answer.

17. A worm-gear has 90 teeth. The velocity ratio is 30 to 1. The circular pitch is $1\frac{1}{2}$ in. The pitch diameter of the worm is 6.00 in. What is the distance on centers? What is the throat diameter of the worm-gear?

CHAPTER XVII

TRAINS OF MECHANISMS

247. Gear Trains. When motion is transmitted from a driver to a driven part of a machine, it is often desirable to use intermediate mechanisms. Although it is theoretically possible to procure any velocity ratio between a driver and a driven member without any intermediate parts, practical considerations may make it necessary with large velocity ratios to use a ***train of mechanisms.***

The commonest train is made up of a series of gears rotating on fixed axes. Rolling cylinders or cones transmitting motion by friction between the rolling surfaces may be substituted for the gears.

In Fig. 245, let the circles represent the pitch circles of the gears, or the friction surfaces of rolling cylinders. The first gear of the train A is called the driver, and the gear B to which it imparts motion is a driven gear. The driven gear B is fixed on the same shaft as the gear C, which is the driver of the pair C

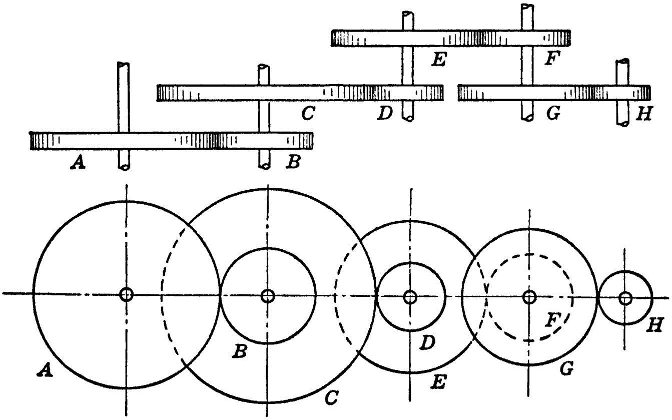


FIG. 245

and D, etc. The velocity ratio between any driver and its driven gear is the inverse ratio of their diameters. Therefore

$$\frac{N_a}{N_b} = \frac{D_b}{D_a}, \qquad \frac{N_c}{N_d} = \frac{D_d}{D_c},$$

where N_a, N_b, etc. denote revolutions of A, B, etc. per unit of time, and D_a, D_b, etc. denote diameters of A, B, etc.

Since B and C are fixed to the same shaft, they have the same angular velocity. Therefore

$$N_b = N_c, \qquad \frac{N_c}{N_a} = \frac{D_a}{D_b},$$

and

$$N_c = N_d \frac{D_d}{D_c}, \qquad \frac{N_d}{N_a} = \frac{D_a}{D_b} \times \frac{D_c}{D_d}.$$

Similarly, we find m, the train value, to be

$$m = \frac{N_h}{N_a} = \frac{D_a}{D_b} \times \frac{D_c}{D_d} \times \frac{D_e}{D_f} \times \frac{D_g}{D_h}. \tag{101}$$

It is apparent that the velocity ratio of the last and the first gear in the train, called the ***train value,*** is equal to the *product of the diameters of all the driving gears divided by the product of the diameters of all the driven gears.* The train value is also the product of the ratios for each individual pair of gears.

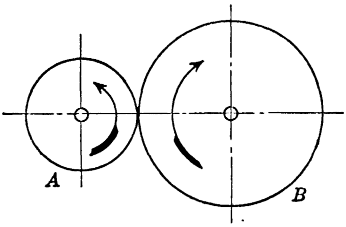


FIG. 246

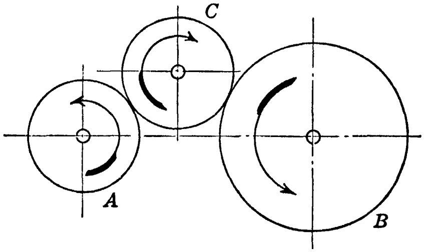


FIG. 247

Idlers. When any gear of the train acts both as a driven and as a driving gear, it is called an idle gear or idler. Such a gear has no effect on the train value except to reverse the direction of rotation of the last driven gear. In Fig. 246, the gears

A and B rotate in opposite directions, and their velocity ratio is

$$\frac{N_b}{N_a} = \frac{D_a}{D_b}.$$

In Fig. 247, a gear C has been inserted between A and B, and these two gears now rotate in the same direction. If m denotes the train value, we have

$$m = \frac{N_b}{N_a} = \frac{D_a}{D_c} \times \frac{D_c}{D_b} = \frac{D_a}{D_b}, \tag{102}$$

which is the same as though the idler were omitted. The main use of the idler is to connect a driving and a driven gear when the center distance is such as to require excessively large gears.

In the case of gears, it is usually easier to count the teeth on the gears than to find the pitch diameters. The numbers of teeth on the gears can be used instead of the gear diameters in calculating train value. For example, in Fig. 248, the gears have diameters and numbers of teeth as follows:

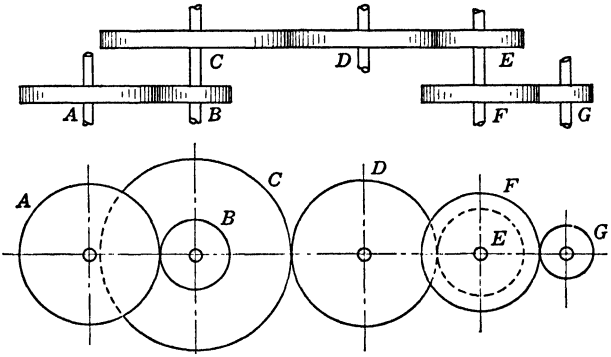


FIG. 248

A	15	inches	diameter	30 teeth
B	7½	"	"	15 "
C	20	"	"	40 "
D	18	"	"	36 "
E	8	"	"	16 "
F	12	"	"	24 "
G	6	"	"	12 "

$$m = \frac{N_g}{N_a} = \frac{D_a}{D_b} \times \frac{D_c}{D_d} \times \frac{D_d}{D_e} \times \frac{D_f}{D_g} = \frac{15 \times 20 \times 18 \times 12}{7.5 \times 18 \times 8 \times 6} = 10,$$

or, using the teeth numbers, we have

$$m = \frac{30 \times 40 \times 36 \times 24}{15 \times 36 \times 16 \times 12} = 10.$$

In this case, G rotates counter-clockwise. If the idler D is removed, and the gears C and E brought into mesh, the train value would still be 10, but G would rotate clockwise.

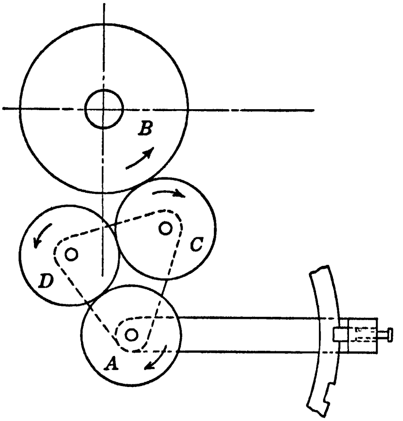


FIG. 249

248. Reversible Gear Trains. When it is necessary to reverse the direction of rotation of the driven shaft, several types of gear trains may be used. A simple arrangement is shown in Fig. 249. Here gear A is the driver, and B the driven. Idler D is always in mesh with A and with idler C, but D and C are carried on a frame pivoted on the shaft of A so that either D or C can be brought into mesh with gear B. In the position shown, B rotates in the direction opposite to A; but when the frame is moved to bring C into mesh with B, the direction of rotation of B is reversed.

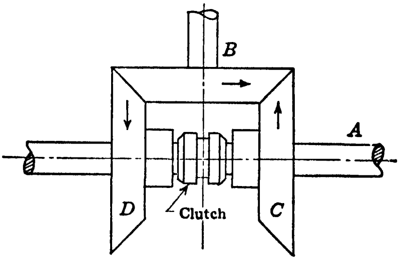


FIG. 250

A form of reversing gear train employing a clutch is shown in Fig. 250. The bevel gears D and C are both in mesh with B, and both are free to rotate on the shaft A. By means of a clutch, either D or C can be made to rotate with the shaft A, the direction of rotation of the gear B depending on whether D or C is driving.

249. Bevel Gears. Bevel gears and friction cones follow the same laws as spur gears or rolling cylinders in the consideration of train values.

250. Pulleys and Belts. Two pulleys connected by a belt or other flexible connector follow the same train laws as gears except that idlers are not used. In Fig. 251, the pulley *A* drives the pulley *B* by means of a belt. Pulley *C* is keyed to the same shaft as *B* and drives pulley *D*. The train value of this belt system is

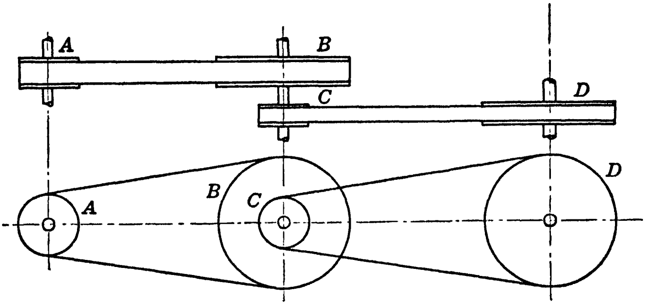


FIG. 251

$$m = \frac{N_d}{N_a} = \frac{D_a}{D_b} \times \frac{D_c}{D_d}.$$

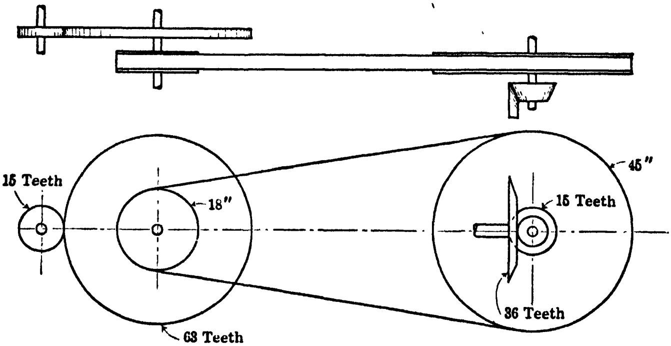


FIG. 252

Gear and belt trains are often combined as in Fig. 252. In this case, the train value is

$$m = \frac{N_d}{N_a} = \frac{15}{63} \times \frac{18}{45} \times \frac{15}{36} = \frac{5}{126}.$$

251. Speed-Change Gears. In many machines, it is desirable to have a constant speed driver such as an electric motor with a series of different speeds on the driven shaft. This may be accomplished by means of sets of sliding gears arranged on the driving and driven shafts. In Fig. 253, the motor M drives the shaft S at a constant speed by means of the gears A and B. On the shaft S are two gears C and D arranged to slide along the shaft so that either can be brought into mesh with its mating gear on the shaft R. When C is moved to the left, it meshes with E and drives the shaft R. When C and D are moved to the right, D meshes with F and drives the shaft R.

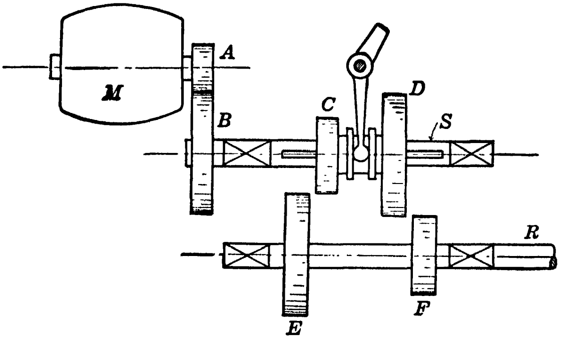


FIG. 253

252. Automobile Transmission. Between the engine and the drive shaft of the automobile, a set of change-gears is placed which usually allows three different speed-ratios between the engine and drive-shaft. In addition to these three speed-ratios, there is also a set of gears to reverse the direction of rotation of the drive-shaft. The transmission gears are arranged as shown in Fig. 254. The gear A is keyed to the engine shaft and drives the countershaft through the gear D. The drive shaft is supported on a bearing in the end of the engine shaft. Gears B and C are splined to the drive shaft and may be moved along the shaft by means of a shifter-yoke and lever. The gears D, E, F, and G rotate as a single unit on the countershaft but may not be moved along it.

When B is moved to the left, the jaw clutch engages with A, and the drive shaft rotates at the same speed as the engine shaft. This is called the ***high-speed position.***

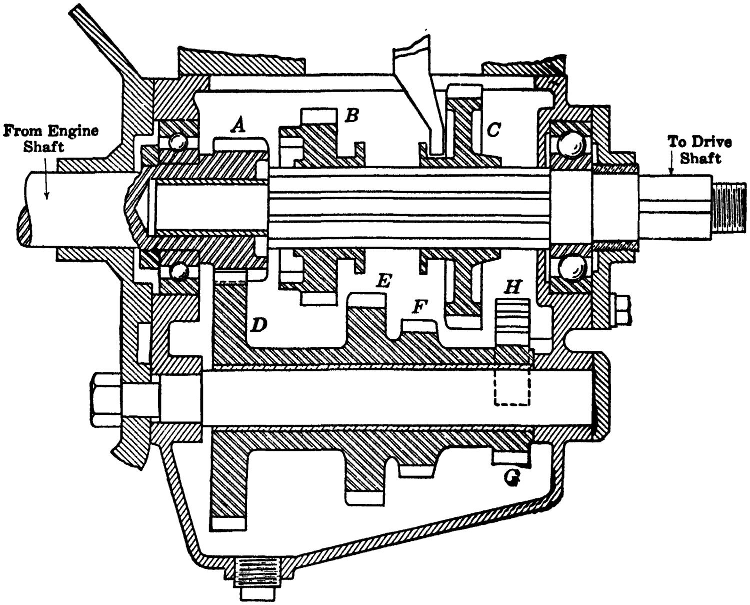


FIG. 254

When B is moved to the right, it engages with and is driven by the gear E on the countershaft. This is ***second-speed position,*** and the velocity ratio is then

$$m = \frac{\text{speed of drive shaft}}{\text{speed of engine shaft}} = \frac{D_a}{D_d} \times \frac{D_e}{D_b}.$$

When B is in the position shown, that is, not engaged with E or the clutch on A, the gear C may be moved to the left and engage with F. This is the ***low-speed position*** and the velocity ratio is

$$m = \frac{\text{speed of drive shaft}}{\text{speed of engine shaft}} = \frac{D_a}{D_d} \times \frac{D_f}{D_c}.$$

When C is moved to the right, it will engage with the idler H, which is driven by the gear G. This will reverse the direction of rotation of the gear C and the drive shaft, and is called ***reverse position.***

Speed-change gearing similar to that just described is used on a variety of machine tools.

253. Lathe Speed Gear-Box. A type of speed-change gear used on modern engine lathes is illustrated in Fig. 255. The driven shaft B carries a nest of gears of varying pitch diameters forming a gear cone. The driving-shaft carries a gear A driven through a feather-key or spline so that the gear can be moved along the shaft. An idler C, in mesh with gear A, is carried on a bracket pivoted on the driving-shaft. If the gear A and the bracket are slid along the driving-shaft, the idler C can be brought into mesh with any of the cone gears D, E, F, G, or H, thus varying the speed of the driven shaft B.

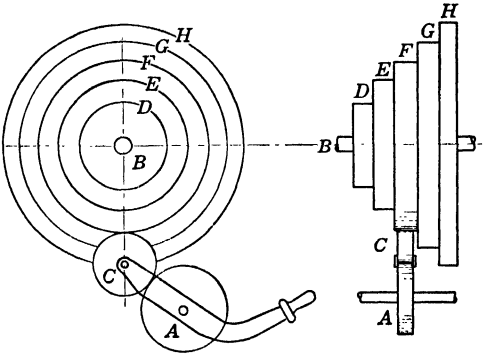


FIG. 255

254. Epicyclic Trains. When the axes of the members of a train of mechanism are carried in a revolving arm, we have an epicyclic train. If the train is a gear train, it is commonly called planetary gearing.

In Fig. 256, two friction wheels, or the pitch lines of two gears, are carried on shafts fixed to an arm A. Assume that wheel B is stationary, and wheel C is free to rotate on its axis C. When A is rotated $\frac{1}{4}$ turn in a clockwise direction, the center C will be in position C_1. Since the surfaces of C and D roll together, the point d on C will come into contact with d_1 on B, since arc

cd_1 is equal to arc cd. The point d_1 is now vertically above the center C_1, and the wheel C has rotated ½ revolution. Similarly

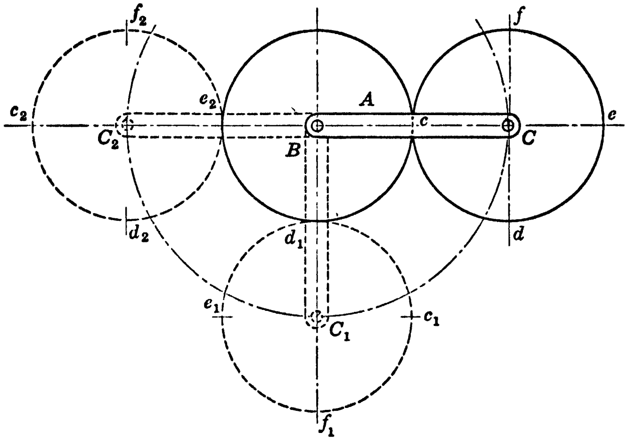


FIG. 256

when the arm has moved ½ revolution to BC_2, the point e on C is in contact with point c_2 on B and d has moved to d_2. The wheel C has, therefore, made one complete revolution. When the arm A has made one complete revolution, d has returned to its original position, but has made two revolutions.

In Fig. 257, the wheel B is n times as large as the wheel C. By an analysis similar to the above, we find that when the arm makes one revolution, the wheel C will turn $n + 1$ revolutions.

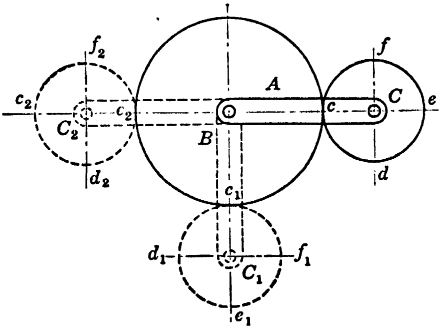


FIG. 257

255. Velocity Ratios in Epicyclic Trains. Consider, in Fig. 258, a train of wheels supported on bearings in the arm A. Between the wheels B and C are arranged any number of gears in a train of which B is the driver and C the driven wheel.

Move the arm A through the angle θ to the position A_1, moving the wheel C to the position C_1. Assume that the gear B is moved by some external means so that b moves to the position b_1, and the train value is such that the movement of B causes C to turn until the point c moves to c_1.

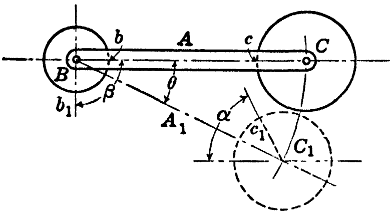


FIG. 258

The absolute angular motions of the arm, the wheel B and the wheel C are represented by the angles θ, β, and α. The angular motion of B relative to the arm is represented by the angle $(\beta - \theta)$, and the angular motion of C relative to the arm is represented by $(\alpha - \theta)$. The train value is the ratio of the angular velocity of C to B when the arm is considered stationary. If m denotes the train value, we may write

$$(103) \qquad m = \text{train value} = \frac{\alpha - \theta}{\beta - \theta}, \quad \text{or} \quad m = \frac{N_c - N_a}{N_b - N_a},$$

where N_a, N_b, and N_c = the revolutions of the arm, wheel B, and wheel C respectively.

The absolute revolutions of any part of the system can be found by transforming this equation as follows:

$$(104) \quad N_a = \frac{N_c - mN_b}{1 - m} = \frac{mN_b - N_c}{m - 1}, \quad N_b = \frac{N_c + (m - 1)N_a}{m},$$

$$(105) \quad N_c = mN_b - (m - 1)N_a = m(N_b - N_a) + N_a.$$

In these equations, the train value, m, is positive when the last wheel, C, revolves in the same direction as the first wheel, B, and negative when they revolve in opposite directions. Also if the wheels rotate in the same direction as the arm, N_b and N_c will be positive. If either of the wheels rotates in the opposite direction from the arm, the sign of N will be negative.

EXAMPLE. As an example of the application of these formulae, consider the epicyclic train in Fig. 259. First consider that the gear B is stationary, and A is rotated clockwise at 10 r.p.m. The train value of the gears is

$$m = \frac{D_b}{D_d} \times \frac{D_d}{D_e} \times \frac{D_e}{D_c} = \frac{20}{80} = +\frac{1}{4} = +0.25,$$

$$N_a = 10, \qquad N_b = 0,$$

$$N_c = m(N_b - N_a) + N_a = 0.25(0 - 10) + 10 = +7.5,$$

whence

$$\frac{N_c}{N_a} = \frac{7.5}{10} = 0.75.$$

Now consider that B is free to rotate, and C is fixed. Then

$$N_a = 10, \quad N_c = 0, \quad m = 0.25,$$

$$N_b = \frac{N_c + (m - 1)N_a}{m} = \frac{0 + (.25 - 1)10}{.25} = -30,$$

whence

$$\frac{N_b}{N_a} = \frac{-30}{10} = -3.$$

Therefore, when the arm rotates once in a clockwise direction, the gear B makes three revolutions in a counter-clockwise direction.

256. Velocity Ratios by Tabulation. In many cases, it is easier and clearer to determine velocity ratios if the gear train is first locked and the arm revolved, and then the arm held stationary and the gears revolved. The following examples will illustrate this principle.

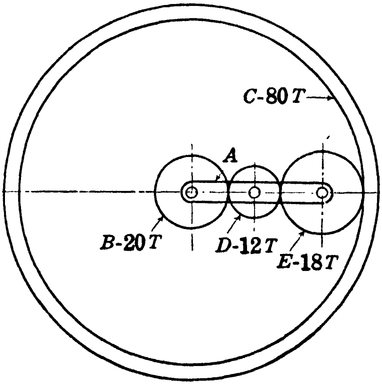


FIG. 259

EXAMPLE 1. In Fig. 259, consider the gear B fixed. The velocity ratio between the gear C and the arm is required. First, lock the gear train, and revolve the arm one revolution clockwise. Each gear makes one complete revolution in a clockwise direction. Tabulate these values. Second, hold the arm stationary and revolve the gear *B* one revolution counter-clockwise to its original position.

	Rev. of *A*	Rev. of *B*	Rev. of *C*	Rev. of *D*	Rev. of *E*
Train locked	+ 1	+ 1	+ 1	+ 1	+ 1
Arm stationary	0	− 1	$-\frac{1}{4}$	$+1\frac{2}{3}$	$-1\frac{1}{9}$
Net revolutions	+ 1	0	$+\frac{3}{4}$	$+2\frac{2}{3}$	$-\frac{1}{9}$

Since *B* has 20 teeth, and *D* has 12 teeth, the gear *D* will rotate $^{20}/_{12}$ or $1^{2}/_{3}$ revolutions clockwise. The gear *E* with 18 teeth will revolve $^{20}/_{18}$ or $1^{1}/_{9}$ revolutions counter-clockwise. The gear *C* will revolve $^{20}/_{80}$ or $^{1}/_{4}$ revolution counter-clockwise. Tabulate these values and add to those tabulated when the wheel train was locked. The sum in each case will be the resultant, or net number, of absolute revolutions of the gears for one revolution of the arm *A*. Note that the gear *C* turned $\frac{3}{4}$ of a revolution clockwise, which agrees with the value obtained by the formula in § 255.

Example 2. Suppose, in Fig. 259, that the gear *C* is held stationary, and the arm revolved. As before, assume that the gear train is locked, and the arm rotated through one revolution clockwise. Then each gear has made one absolute revolution. Tabulate these. Now hold the arm stationary, and rotate the gear *C* counter-clockwise one revolution to its original position.

	Rev. *A*	Rev. *B*	Rev. *C*	Rev. *D*	Rev. *E*
Train locked	+ 1	+ 1	+ 1	+ 1	+ 1
Arm stationary	0	− 4	− 1	$6\frac{2}{3}$	$-4\frac{4}{9}$
Net revolutions	+ 1	− 3	0	$7\frac{2}{3}$	$-3\frac{4}{9}$

Since the gear *C* has 80 teeth, and the gear *E* has 18 teeth, the gear *E* will rotate $^{80}/_{18}$, or $4^{4}/_{9}$ revolutions counter-clockwise. The gear *B* with 20 teeth will rotate $^{80}/_{20}$ or 4 revolutions counter-clockwise. The net result will be that *B* rotates − 3 revolutions, or 3 revolutions counter-clockwise, for one revolution of the arm clockwise. This checks the result obtained by the formula in § 255.

The tabulation method can be readily applied to any complicated train of epicyclic gearing.

257. Triplex Hoist. A good example of an epicyclic gear train is shown in Fig. 260. The gear B is the driver, and is rotated by means of a chain on the chain-sheave M. The gears D and E are carried on the arm A, which is keyed to a sleeve which also carries the chain-sheave K. The gear E meshes with the annular C, which, being a part of the frame, is stationary. The train value of the epicyclic train consisting of the gears B, D, E, and C and the arm A is

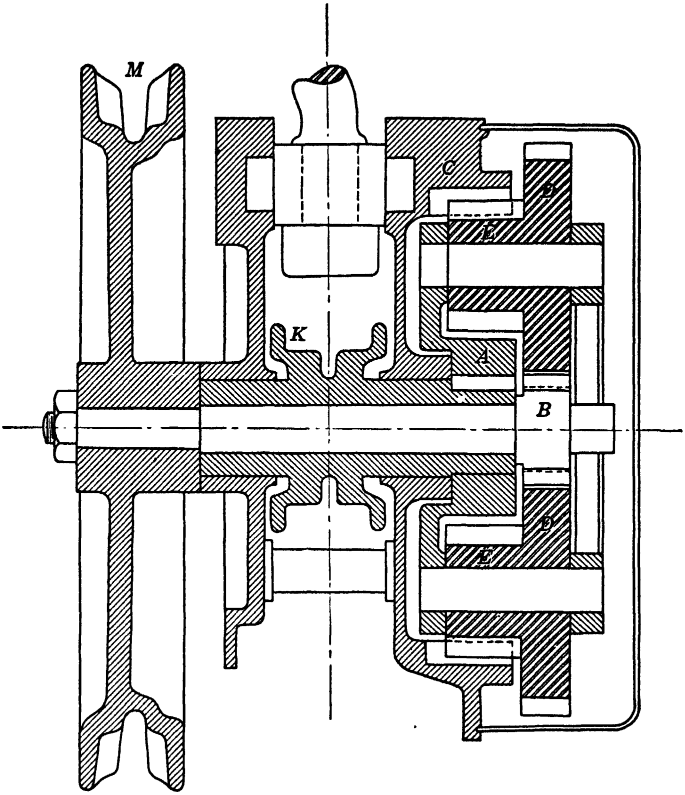


FIG. 260

$$m = \frac{D_b}{D_d} \times \frac{D_e}{D_c} = \frac{12}{29} \times \frac{14}{55} = +\ 0.1053.$$

Assuming the gear B to be turned one revolution by means of the hand chain and sheave M, we find the revolutions of the arm and hoisting chain-sheave K by means of equation

$$N_a = \frac{mN_b - N_c}{m - 1} = \frac{0.1053 \times 1 - 0}{0.1053 - 1} = -0.1178.$$

258. Differential Gear. In order to drive the rear wheels of an automobile or tractor from a single engine shaft and still to allow for different speeds in the wheels when turning curves, a differential similar to Fig. 261 is used.

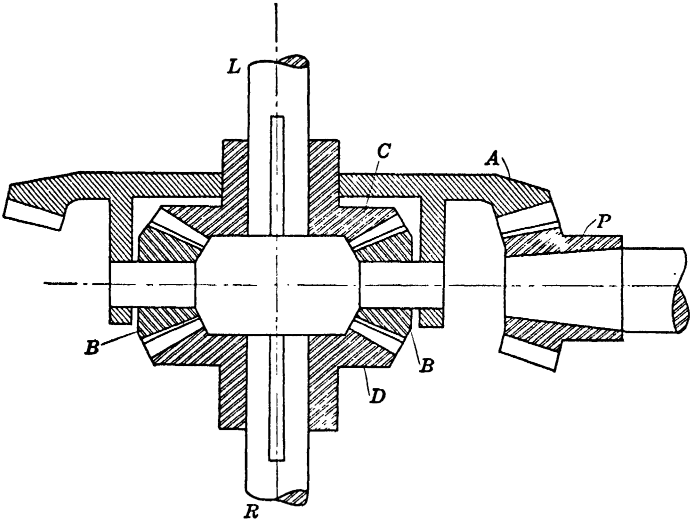


FIG. 261

Power is delivered from the engine through the shaft and gear P. The gear P drives the gear A which carries the bearings for the bevel gears B on arms. There are two, three, or four gears B, arranged to distribute the load around the gears C and D. The gears B mesh with both C and D. The gear C is keyed to the shaft L which drives the left wheel. The gear D is keyed to the shaft R which drives the right wheel.

When the machine is traveling on a straight road, shafts L and R, and the gears C and D, rotate at the same speed as a single unit, the gears B simply acting as connectors between the gear A and the gears C and D, and not revolving on their own axes.

When turning a curve, say to the right, the left wheel must travel faster than the right wheel. This causes the gear *C* to rotate faster than *D*. The gears *B* then turn on their own axes while the gear *A* rotates at a speed which is the mean of the speeds of *C* and *D*. If one wheel becomes locked, then the other wheel will rotate at twice the speed of the gear *A*.

259. Rope-Winding Machine. An interesting application of epicycloidal gear trains is used in certain kinds of rope-winding machinery. Figure 262 shows the gear train. The gears *B*, *C*, and *D* are carried on a rotating frame, *H*, and each gear carries a spool, or bobbin, *S*, from which a strand of hemp or wire is fed to form the rope. As the frame revolves, twisting the strands together to form a rope, the strands themselves should not twist, or should be given a slight twist in the opposite direction. If the gear *A* is fixed, and gears *A*, *B*, *C*, and *D* are equal in size, there will be no rotation of *B*, *C*, and *D* as the frame revolves; and the strands will not be individually twisted although twisted together. If *B*, *C*, and *D* are slightly larger than *A*, then *B*, *C*, and *D* will revolve slowly in a direction opposite to the revolution of the frame, giving each strand a slight individual twist at the same time that they are being twisted together.

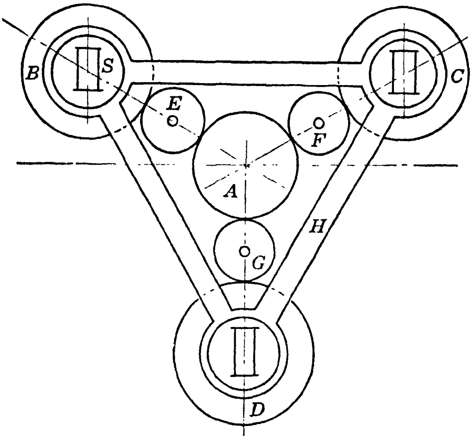
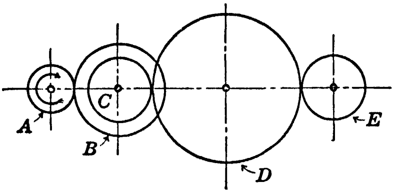


FIG. 262

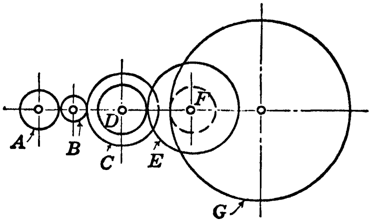
260. Practical Applications of Epicyclic Trains. In addition to the uses mentioned above, epicyclic gear trains are used in such mechanisms as the Ford Planetary transmission for the model T automobile, speed-reducers, hoists, differential or compensating devices, and similar mechanisms.

PROBLEMS

1. In the gear train shown, determine the train value, or ratio of the angular velocity of the last gear to that of the driving gear A. The number of teeth on the gears are, $A = 15$, $B = 30$, $C = 20$, $D = 45$, and $E = 20$.

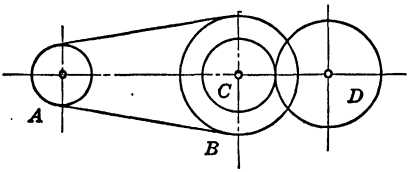


Problem 1



Problem 2

2. Determine the angular velocity of G if A rotates at 100 r.p.m. The gear diameters are, $A = 6$ in., $B = 4$ in., $C = 12$ in., $D = 8$ in., $E = 15$ in., $F = 8$ in., $G = 30$ in.



Problem 3

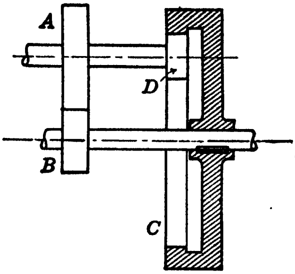
3. If A rotates at 60 r.p.m., determine the speed of D. The diameter of A is 10 in. and of B 20 in. The gear C has 30 teeth and the gear D 45 teeth.

4. A shaft rotating clockwise is to act as the driver for a parallel shaft 6 in. away, which is also rotating clockwise. Show by sketches how to do this, using (a) two gears, (b) three gears, and (c) a belt, using a 4-in. pulley as the driver. If the driver rotates 100 r.p.m., and the driven 200 r.p.m., give the gear or pulley diameters in each case.

5. Find the number of teeth for a gear train that drives a shaft at a speed of 102 r.p.m. from a shaft making 6 r.p.m., if no gear of less than 12 teeth nor more than 60 teeth is used. Show this train.

6. In a two-speed gear drive, the shafts are 10 in. apart. The driving shaft is to run at 250 r.p.m. and the driven at 50 and 75 r.p.m. Sketch this drive and determine the diameters of all gears.

7. In the gear drive shown, *A* is fixed to the shaft and *D* is free to rotate on the shaft. Gears *B* and *C* are fixed to the same shaft. The gear *A* has 24 teeth, *B* has 16 teeth, *C* has 52 teeth and *D* has 12. If *A* drives and rotates at 100 r.p.m., determine the speed of *D* and the speed of *D* relative to the driving shaft.



PROBLEM 7

8. A train of gears consists of a 30-tooth gear *A* in mesh with a 20-tooth gear *B*, which in turn meshes with a 16-tooth gear *C*. These gears *A*, *B*, and *C* are mounted on an arm whose axis of rotation is at the center of the gear *A*. Keyed to the gear *C* is a 32-tooth gear meshing with an annular gear whose center of rotation is at the center of *A*.

(a) How many teeth has the annular?

(b) If *A* is rotated through 40 revolutions clockwise while the annular is held stationary, how many revolutions does the arm make?

(c) If *A* is held stationary, and the annular is free to rotate, how many revolutions will the annular make when the arm is rotated one revolution clockwise?

9. Given a train of gears consisting of a 30-tooth, 5-pitch pinion concentric with an annular gear and having three equal idler gears, each meshing with the pinion and annular.

(a) With a speed reduction of 4, what size annular and idlers are required if the idler axes are fixed?

(b) What reduction will be obtained if there are 120 teeth on the annular, if the annular is held stationary, and if the frame supporting the idlers rotates?

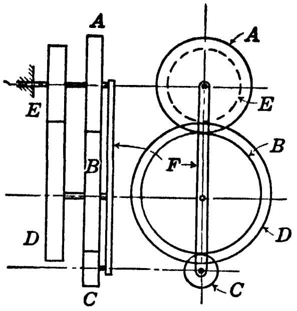
10. A speed reducer built by the Foote Bros. Gear and Machine Co. consists of a driving pinion of 12 teeth driving 3 pinions of 30 teeth each, rotating on fixed centers arranged on a circle concentric with the driving pinion axis. The 30-tooth pinions drive an annular gear on the driven shaft.

(a) How many teeth has the annular?

(b) What is the ratio of speed reduction between the driving and driven shafts?

11. Gears *A*, *B*, *C*, and *D* are free to rotate individually on a single shaft *S*. A second shaft is mounted on arms which may also rotate about the axis of the shaft *S*. This second shaft carries a solid nest of gears *E*, *F*, *G*, and *H* in mesh with *A*, *B*, *C*, and *D* respectively.

The teeth on these gears are as follows: $A = 15$, $B = 10$, $C = 20$, $D = 30$, $E = 35$, $F = 40$, $G = 30$, $H = 20$.



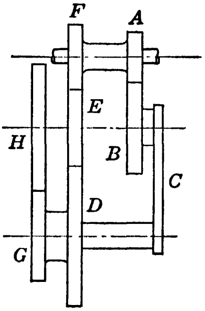
PROBLEM 12

(a) If *A* is held stationary and the arm rotated once, how many revolutions will *B*, *C*, and *D* make?

(b) If *B* is held stationary and *A* is rotated once, how many turns will the arm make?

12. In the figure, the arm *F* rotates about the center of the gear *A*. Gear *A* has 20 teeth, *B* has 25 teeth, *C* has 15 teeth, *D* has 30 teeth, and *E* has 15 teeth. Gears *B* and *D* are fixed to the same shaft, but gears *A* and *E* are free to turn on the shaft.

If *E* is held stationary and *A* rotates at 12 r.p.m., at what speed will the arm rotate?



PROBLEM 14

13. In Problem 12, if the gear *A* is held stationary, and gear *C* makes 3 r.p.m. relative to the arm *F*, determine the angular velocity of gear *E* in r.p.m.

14. In the figure, gears *A* and *F* are fixed to the same shaft. The shafts through *A* and *B* are held on stationary bearings. The arm *C* and the gears *B*, *D*, *E*, *G*, and *H* can turn freely on their shafts. The arm *C* and the gear *B* turn as one piece, and the gears *D* and *G* turn as one piece. Gear *A* has 12 teeth, *B* has 24 teeth, *D* has 36 teeth, *E* has 20 teeth, *F* has 16 teeth, *G* has 24 teeth, and *H* has 32 teeth. When the shaft carrying *A* and *F* rotates at a speed of 10 r.p.m., what will be the angular velocity of *H*?

CHAPTER XVIII

MISCELLANEOUS MECHANISMS

261. Mechanisms for Special Purposes. There are many types of mechanisms developed for use in special cases to meet the numerous requirements of modern machinery. Some of these are simple, while others appear to be very complicated; but a careful analysis will show that they can all be grouped according to the classifications of mechanisms discussed in preceding chapters. To describe and to discuss all of these special cases would require many volumes; therefore the discussion here will be confined to a few common types which will serve to illustrate what can be accomplished.

262. Intermittent Motion. In many machines such as automatic machine-tools, wire-bending machines, mechanical stokers, etc., many of the parts are in motion only part of the time; but the periods of motion and of rest must have a definite relation to the movement of the machine as a whole. Motions of this kind are called intermittent motions. Intermittent motions may be produced by the use of ratchets, escapements, cams, intermittent gearing, and similar devices.

263. Ratchets. A simple device for producing intermittent motion is the ***ratchet wheel*** and ***pawl,*** or simply ratchet. The ratchet consists of a wheel having notches or teeth cut in its rim, into which fits a pawl carried on an oscillating or reciprocating member. The teeth and pawl are so shaped that during the forward stroke the pawl engages with the teeth of the ratchet and moves the ratchet wheel through a part of a revolution. On the return stroke, the pawl simply moves over the teeth without turning the wheel. To prevent the ratchet wheel from turning backwards, a ***detent pawl*** attached to a stationary member is used.

Figure 263 shows a simple ratchet and pawl arranged for use in operating the feed screw of a shaper or other machine tool. As the arm on lever *A* rotates counter-clockwise, *C* is first moved to the left; the pawl *P* pivoted to the arm *C* at *p* engages with the teeth in the ratchet wheel at *n*, causing a counter-clockwise rotation. As *A* continues to rotate, *p* reaches its extreme position to the left and moves back toward the right, the pawl sliding over the top of the teeth and producing no movement of the ratchet wheel. The angular movement of *C*, and therefore the angular movement of the ratchet wheel, is varied by changing the length of the driving arm *A*. A detent pawl to prevent backward movement of the ratchet wheel is shown at *E*.

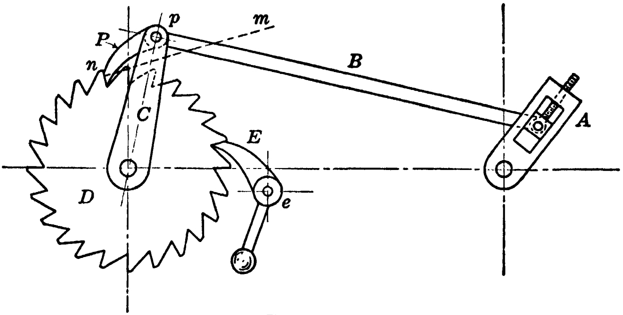


Fig. 263

The shape of the teeth and of the pawl should be such that the normal *mn* to the contact surfaces passes between *p*, the pivot point of the pawl, and the center of the ratchet wheel to prevent the pawl's slipping out of engagement. If the ratchet and pawl are not given the correct shape, it is necessary to provide a weight or spring to hold the pawl in engagement.

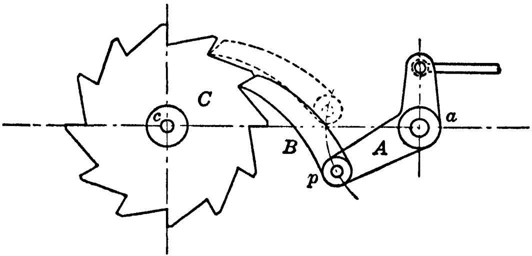
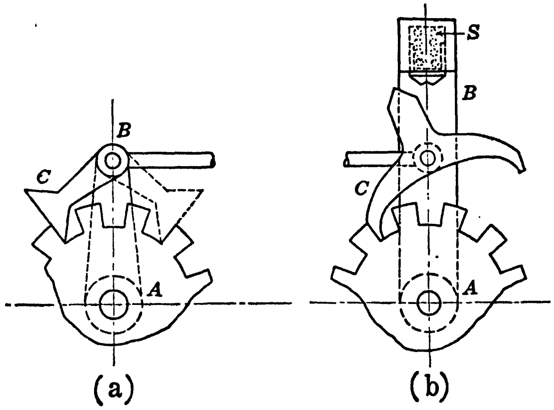


Fig. 264

When the pawl is not made to move concentric to the ratchet wheel, the pawl and ratchet tooth will have relative motion during engagement, which causes slip and wear, and may cause disengagement. Figure 264 illustrates this. To reduce this relative movement to a minimum, the pawl pin *p* should swing equally on both sides of the center-line *ac*, and the pawl *B* should be as nearly normal to *ac* as possible.



(a) (b)

FIG. 265

In feed mechanisms on machine tools, ***reversible ratchets*** are desirable, and two mechanisms of this type are shown in Fig. 265. A spring *S* is used in one of the mechanisms to keep the pawl in engagement during either stroke of the arm, or to hold it entirely out of engagement.

Figure 266 shows an arrangement by which the ratchet wheel is driven forward during both strokes or oscillations of the bell crank *A*. A detent pawl is not required with this arrangement.

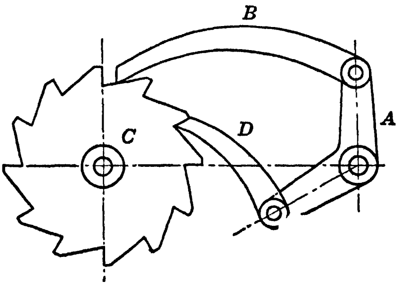


FIG. 266

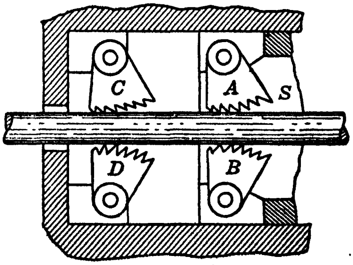


FIG. 267

A type of ratchet used for feeding wire or small bar-stock to machine tools is illustrated in Fig. 267. When the sliding member *S* moves to the right, the ratchet segments *A* and *B* grip the wire and feed it forward. When the slide *S* moves to the left, the segments *A* and *B* slide over the wire, which is prevented from moving backwards by the detent ratchet segments *C* and *D*.

Many other forms of ratchets are in use, some giving positive action, some depending on friction instead of teeth and pawls. Those depending on friction are sometimes called ***silent ratchets*** and are used where noise is objectionable, and where positive action is not absolutely required. Various forms are shown in Fig. 268. Note that the free-wheeling device on the modern automobile is a modification of Fig. 268c.

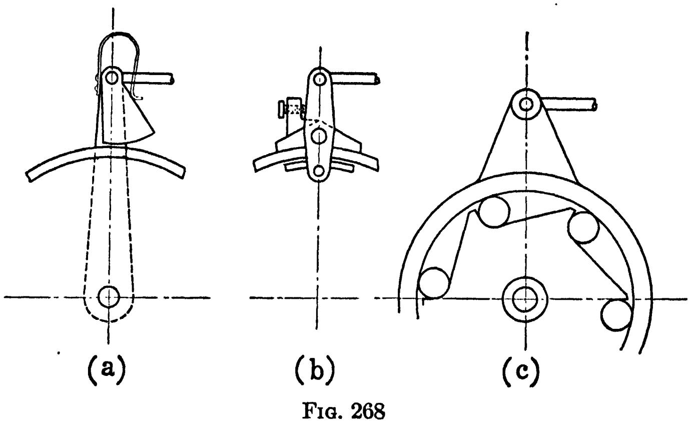

(a) (b) (c)

FIG. 268

264. Escapements. An escapement is a ratchet mechanism in which the driven member or follower is automatically and continuously stopped and released. It is used where alternate periods of rest and motion are required with a definite forward movement of the driven member for each movement period. The most common usage is in clocks and similar timing devices, to allow the driving springs or weights to act intermittently, giving the pendulum, or balance wheel, a definite impulse during each period of action.

Figure 269 shows a form of escapement used with pendulum clocks. Power from a spring or from weights tends to turn the toothed wheel to the right. Above the wheel is a double pawl *ABC* pivoted at *B* and having beveled ends at *A* and *C*, called ***pallets.*** The angles *BDE* and *BFK* should be equal. The pendulum is pivoted on the same axis as the pawl, that is, at *B*, and as it swings, causes the pawl to oscillate. As the pendulum swings to the left, carrying the pawl with it, the point *P* of a tooth slides along the surface *AD*. As *AD* is formed by a circular arc whose center is at point *B*, there will be no movement of the wheel, and no impulse imparted to the pawl. When the pawl has moved so that the tooth-point *P* is in contact with the pallet surface *DE*, the wheel will rotate to the right, exerting a pressure perpendicular to the pallet surface *DE*, imparting a rotary impulse to the pawl *ABC*, which in turn imparts the impulse to the pendulum. As *P* leaves the surface *DE* at *E*, the point of a tooth at *R* moves forward to make contact with the surface *FH*, which is also a circular arc with point *B* as the center. The wheel is thus held stationary while the pendulum and pawl complete their swing, point *R* moving along *FH*. When the pendulum swings back, the tooth-point *R* moves back to *F* and along the pallet surface *FG* again, allowing the wheel to move forward and impart a second impulse to the pawl and pendulum. As point *R* leaves the pallet *FG* at *G*, another tooth-point moves to contact with the surface *AD*, and the cycle is repeated.

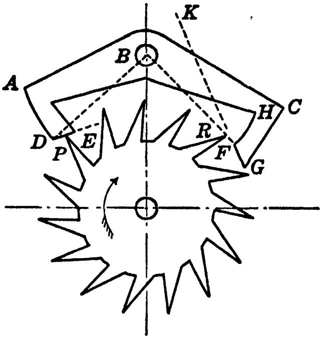


FIG. 269

The natural period of vibration of the pendulum depends on its weight and length, and its weight is such that its period of vibration is not affected by the small impulses received through the pallets. The impulses are sufficient, however, to overcome the retarding effect of air resistance and friction, so that the amplitude of the pendulum vibrations is maintained, as long as the power spring or weights can supply the slight impulse.

Figure 270 shows a lever escapement used in watches or in clocks using a balance wheel instead of a pendulum. In this escapement, the pallets *DE* and *FG* are usually jewels carried in the oscillating pawl and lever *ABCM* pivoted at *B*. The balance wheel is supported on the balance staff *O*. On the balance staff is a disk carrying a pin *P*. The lever *BM* is slotted at the end to engage with this pin and also carries a pin at *M*, which engages with the edge of the disk.

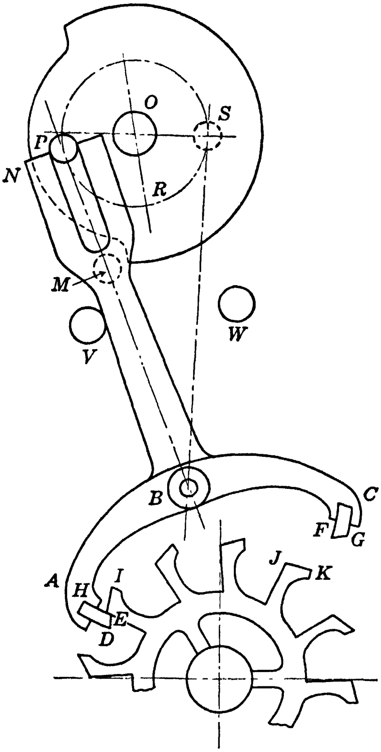


FIG. 270

As the balance wheel and staff oscillate, moving the pin *P* along the path *PRS*, the pin moves in the slotted end of the lever *BM*, moving the lever to the right. The edge of the disk is cut away so as to clear the pin *M* during this movement. At *S* the pin *P* leaves the slotted lever, which is prevented from returning by the pin *M* in contact with the disk edge. When the balance wheel completes its swing, the pin *P* returns, reentering the slot at *S* and moving the lever *BM* to its original position to the left. At *V* and *W* are pins to prevent overswing of the lever *BM*. During the swing of the lever to the right, the escapement wheel is allowed to move, imparting an impulse to the balance wheel through the tooth surface *HI* and the pallet *DE*. On the return swing, the impulse is imparted from the tooth surface *JK* to the pallet *FG*.

265. Intermittent Gearing. Intermittent gearing is a mechanism in which the driver has constant angular velocity, while the follower has alternate periods of motion and rest. Examples of this type of motion are found in revolution counters, simple counting or adding machines, feeding devices, and various types of automatic machinery.

To make the motions positive, some form of tooth-gearing or pin-gearing is used during the driving portion of the cycle; and some form of positive lock is used to hold the follower stationary during its period of rest.

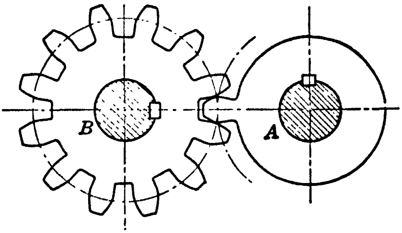


FIG. 271

In Fig. 271 is shown an intermittent motion consisting of a pair of equal diameter gears having teeth cut according to any common system. On the driver all the teeth except one have been cut away, and the follower moves forward only when this tooth is in contact with a tooth on the follower. To insure proper action, the teeth on the follower must be so spaced that, when at rest, one of the tooth spaces is in position to allow the driving tooth to enter without interference. Also, the gears are modified to form locking arcs, as shown in Fig. 272, to prevent relative motion during the rest period.

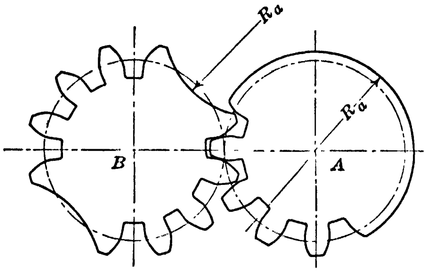
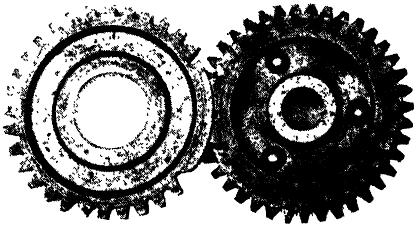


FIG. 272

If the follower is to have a longer period of motion than one tooth of the driver will give, several teeth can be used as shown in Figs. 273 and 274. In Fig. 274, a pair of pins and spurs have been added so that the follower will be gradually accelerated from rest to the angu-



Hartford Special Machinery Co., Hartford, Conn.
FIG. 273. INTERMITTENT GEARS.

lar velocity required during the motion period. This relieves the shock incidental to sudden stopping and starting when the teeth or locking arcs come into action to start or stop the follower. In this figure the locking arc of the driver is larger than the pitch diameter. When the pin P strikes the spur MN, the follower starts to move and accelerates until the tooth T comes in contact with R. The pin P then ceases contact with the spur MN. The locking arc DE must be rounded off to allow the follower to turn while the pin is acting on the spur MN. A second pin and spur is used to decelerate the follower when the driving teeth cease contact. Note that the follower makes one or more revolutions for each revolution of the driver, but that the periods of motion and rest can be varied by increasing or decreasing the length of the locking arc, while the length of the toothed arc remains constant.

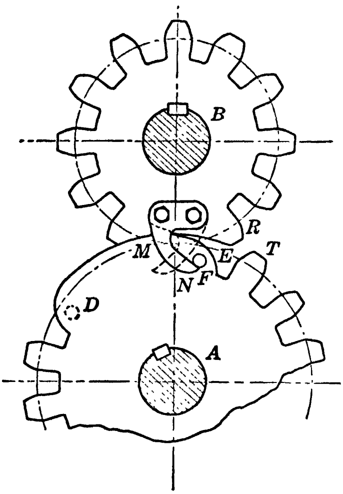


FIG. 274

A simple and durable form of intermittent gearing called the *Geneva wheel* is shown in Fig. 275. The driver is a cylindrical pin, P, rotating about the center A, and engaging with slots cut

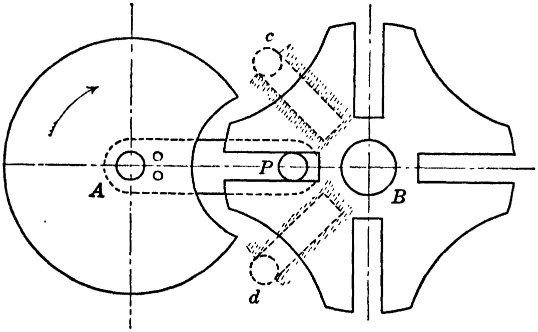
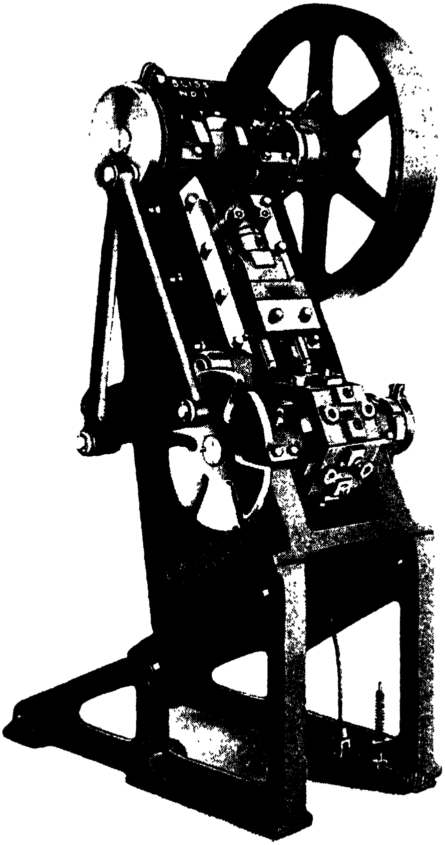


FIG. 275

in a disk or ***star-wheel*** rotating about the center *B*. As the pin *P* enters the slot at *c*, the wheel *B* rotates until the pin leaves the slot at *d*. The wheel *B* then remains stationary while the pin *P* completes its revolution and enters the next slot at *c*. To prevent motion of *B* while *P* is moving from *d* to *c*, a disk is attached to the arm *AP*, and the wheel *B* is cut away between slots to fit the disk. To allow the wheel *B* to turn when the pin *P* enters the slots, part of the disk is cut away at the pin to form clearance for the tips of the wheel *B*.

When the pin first enters the slot, its motion is along the slot and toward *B*. After entering the slot, the pin causes the wheel *B* to have gradually increasing angular velocity until *P* is on the center-line *AB*, after which the angular velocity of *B* gradually decreases. This eliminates the shock due to sudden starting and stopping.

The Geneva motion, or star-wheel, is used in automatic machinery as an indexing device, in motion picture projection machines alternately to feed the film and to stop the picture frames in front of the light aperture, and in many similar devices. A modified form is shown in Fig. 276, where it is used to index the intermittently revolving die-block on a press.



E. W. Bliss Co., Brooklyn, N. Y.

FIG. 276. MODIFIED GENEVA-MOTION USED AS A FEED DEVICE ON A BOLT TRIMMING-PRESS.

If the star-wheel is arranged in a train of wheels each having ten slots, a simple counting machine is produced as shown in Fig. 277. When A makes ten revolutions, B makes one revolution, and when A makes one hundred revolutions, B makes ten revolutions and C one revolution, etc.

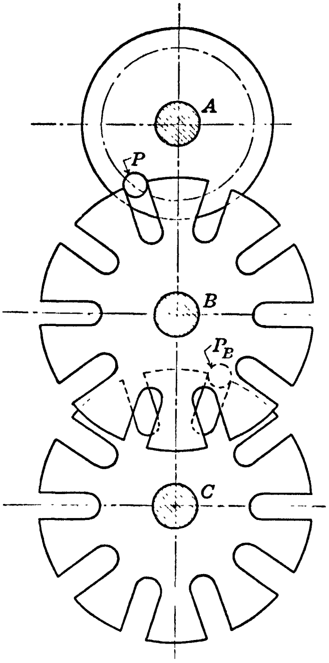
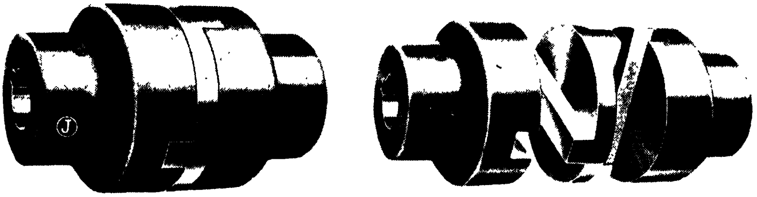


FIG. 277

Many other forms of intermittent gears are in use, but detailed descriptions would require more space than their importance warrants.

266. Oldham's Coupling. The Oldham's coupling is a device used to connect two shafts which are parallel and nearly in line. Usually, it consists of three disks, two of which have a slot cut along a diameter. These slotted disks are attached to the shafts as shown in Fig. 278. The third, or intermediate, disk has two tongues, or projections, on opposite sides of the disk along diameters at right angles to each other which fit into the slots of the mating disks.



W. A. Jones Foundry and Machine Co., Chicago, Ill.

(a) (b)

FIG. 278. OLDHAM-TYPE COUPLING.

Another form of the same device consists of a cross sliding in forks attached to the shafts as in Fig. 279. The velocity ratio transmitted from shaft A to shaft B is a constant, but there is considerable sliding between the tongues and slots, or the cross and forks. Considering a section taken at right angles to the shafts as in Fig. 280, it is readily seen that the branches of the cross must always pass through the centers A and B; and since they are at right angles, the center point of the cross C must travel in a circle of diameter AB. During one-half revolution, the point C moves a distance BA relative to the guiding forks.

The velocity ratio between the shafts is constant so that if the driver is rotating uniformly, the driven shaft also rotates uniformly.

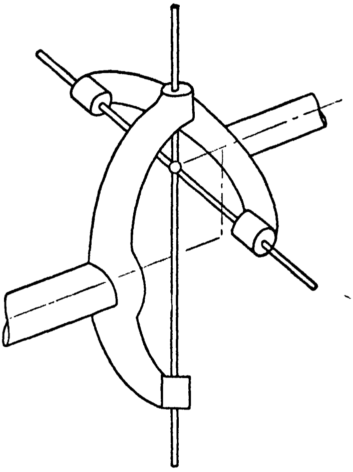


Fig. 279

267. Universal Joint. If the shafts are not parallel but intersect, a cross and forks may be used to connect the shafts, as in Fig. 281. This is the simplest form of the so-called ***universal joint,*** or ***Hooke's coupling,*** other forms being shown in Figs. 282, 283, and 284.

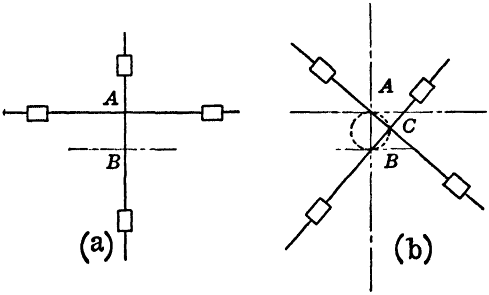


Fig. 280

In the case of two intersecting shafts connected by a universal coupling, the angular velocity ratio is not constant. Although the driven shaft makes one complete revolution for each revolution of the driver, its angular velocity is varying above and below the angular velocity of the driver.

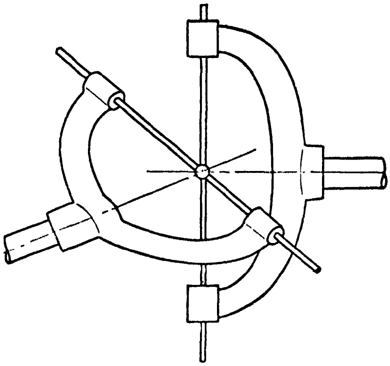
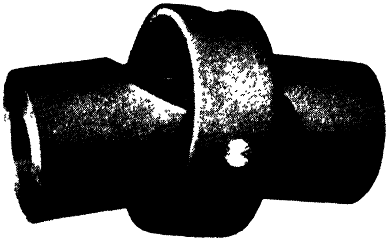
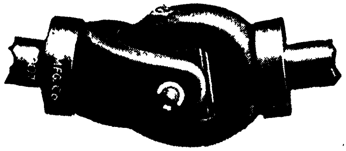


FIG. 281



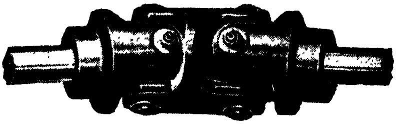
W. A. Jones Foundry and Machine Co., Chicago, Ill.

FIG. 282. RING-TYPE UNIVERSAL JOINT.



Stephens-Adamson Mfg. Co., Aurora, Ill.

FIG. 283. UNIVERSAL JOINT.



Dodge Mfg. Co., Mishawaka, Ind.

FIG. 284. DOUBLE UNIVERSAL JOINT.

Consider Fig. 285, in which a shaft A is driving shaft B through a universal coupling. Shaft B intersects shaft A at an angle α. Place the arms of the cross so that one arm and the forks on shaft A are in the position aOm. Then the other arm connected to the forks on shaft B is in the position b_2Ok. When A rotates through an angle θ so that the fork end a moves to a_1, the shaft B rotates through an angle ϕ, the fork end b moving to position

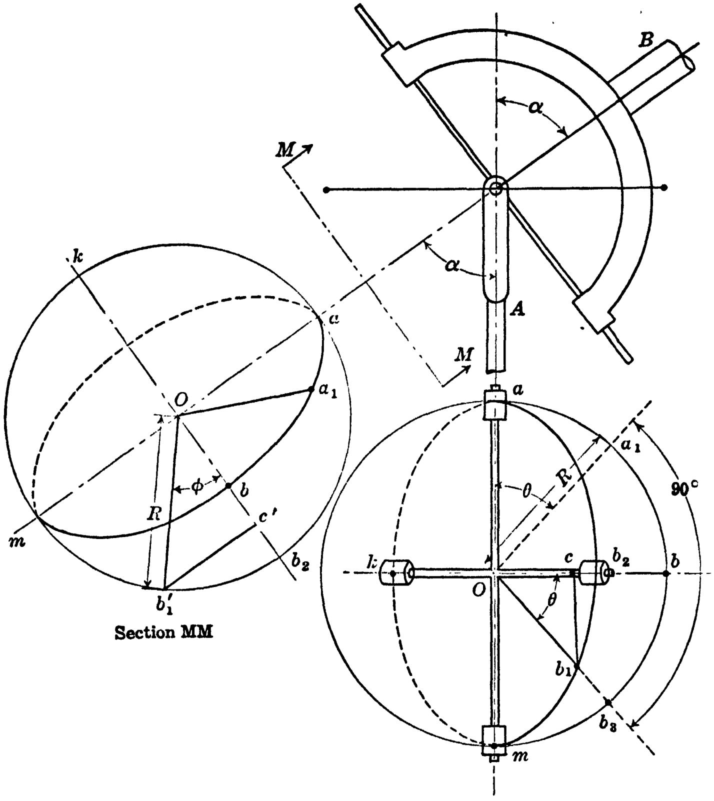


FIG. 285

b_1. To study the relation between the movements θ and ϕ, assume that the axes of the shafts A and B coincide while a moves to a_1; and that the shaft B is then moved through the angle α. When the axes are in line, if a moves from a to a_1, b moves from b to b_3. If the shaft B is now moved sideways through the angle α, the cross will rotate on Oa_1, moving b_3 to b_1 along radial line Ob_3. This is equivalent to starting with the shafts at an angle of α with ends of the cross at a and b_2, then rotating the shaft A until a moves to a_1 and b moves from b_2 to b_1.

The true angle which Ob_1 makes with its original position Ob_2 is shown in section MM as ϕ. In the end view, the projection of this angle on the plane of the paper is the angle $b_1Ob_2 = \theta = a_1Oa$.

From the figure,

$$\sin\theta = \frac{b_1c}{Ob_1}, \quad \sin\phi = \frac{b_1'c'}{Ob_1'} = \frac{b_1c}{R}, \qquad R\cos\phi = Oc' = \frac{Oc}{\cos\alpha}.$$

Substituting for R in the equation for $\sin\phi$, we find

$$\sin\phi = \frac{b_1c\cos\alpha\cos\phi}{Oc}.$$

But

$$\frac{b_1c}{Oc} = \tan\theta. \tag{106}$$

Hence

$$\sin\phi = \tan\theta\cos\alpha\cos\phi, \quad \tan\phi = \tan\theta\cos\alpha,$$

which is the displacement equation for shaft B in terms of the displacement of shaft A and the angle between the shafts, α.

Differentiating with respect to time, we find the angular velocity ratio as follows:

$$\frac{d(\tan\phi)}{dt} = \frac{d(\tan\theta\cos\alpha)}{dt}, \qquad \frac{1}{\cos^2\phi}\frac{d\phi}{dt} = \frac{\cos\alpha}{\cos^2\theta}\frac{d\theta}{dt},$$

$$\frac{\omega_b}{\cos^2\phi} = \omega_a\frac{\cos\alpha}{\cos^2\theta}, \qquad \frac{\omega_b}{\omega_a} = \frac{\cos\alpha\cos^2\phi}{\cos^2\theta} = \frac{\cos\alpha}{\sec^2\phi\cos^2\theta}.$$

But we know that

$$\sec^2\phi = 1 + \tan^2\phi = 1 + \tan^2\theta\cos^2\alpha.$$

Hence

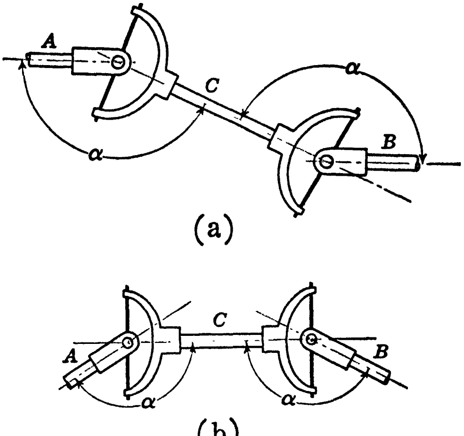
$$\frac{\omega_b}{\omega_a} = \frac{\cos\alpha}{(1+\tan^2\theta\cos^2\alpha)\cos^2\theta} = \frac{\cos\alpha}{\cos^2\theta + \sin^2\theta\cos^2\alpha},$$

or

$$\frac{\omega_b}{\omega_a} = \frac{\cos\alpha}{1 - \sin^2\theta\sin^2\alpha}. \tag{107}$$

This is the equation for velocity ratio in terms of the angle θ through which the driving shaft has moved and the angle α between the shafts.

When α is equal to zero, the velocity ratio becomes 1. When θ is equal to 0, π, 2π, etc., the velocity ratio becomes a minimum, and when θ is equal to $\pi/2$, $3\pi/2$, etc., the velocity ratio becomes a maximum. This shows that if the driving shaft A rotates uniformly, the driven shaft B varies from a minimum to a maximum angular velocity twice during each revolution.



(a)

(b)

FIG. 286

If it is necessary that the driven shaft rotate uniformly, this result may be accomplished by using two universal joints arranged as in Fig. 286, where the intermediate shaft C makes equal angles with the driver A and the driven B. When the shaft A turns through an angle θ, shafts C and B turn through angles γ and ϕ respectively. From the preceding equations:

$$\frac{\omega_c}{\omega_a} = \frac{\cos\alpha}{1 - \sin^2\theta\sin^2\alpha},$$

and

$$\frac{\omega_b}{\omega_c} = \frac{\cos\alpha}{1 - \sin^2(\gamma + 90)\sin^2\alpha}.$$

Combining these equations, we find

$$\frac{\omega_b}{\omega_a} = \frac{\cos^2\alpha}{(1 - \sin^2\theta\sin^2\alpha)(1 - \cos^2\gamma\sin^2\alpha)}; \tag{108}$$

but we know that

$$\tan\gamma = \tan\theta\cos\alpha,$$

$$\cos^2\gamma = \frac{1}{1 + \tan^2\gamma} = \frac{1}{1 + \tan^2\theta\cos^2\alpha}.$$

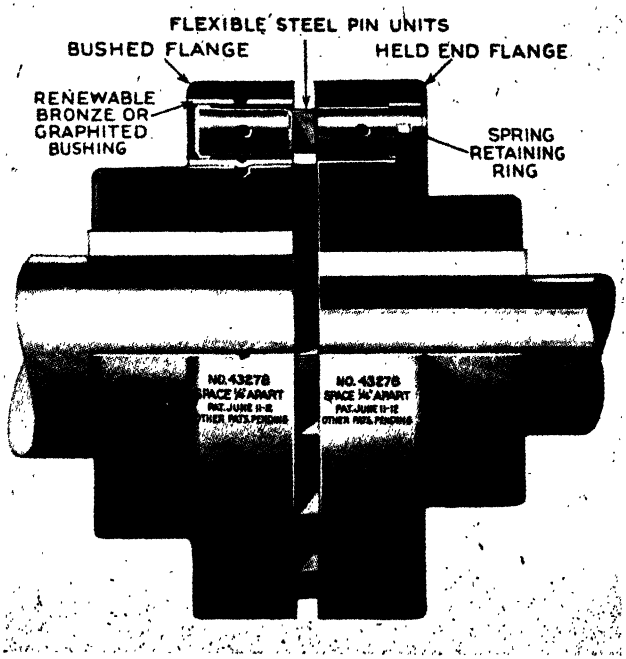
Substituting for $\cos^2 \gamma$ and making the proper trigonometric substitutions, we get

$$\frac{\omega_b}{\omega_a} = 1, \tag{109}$$

which shows that if A rotates with uniform angular velocity, then B also rotates with uniform angular velocity.

268. Flexible Couplings. In addition to the Oldham coupling and the universal joint, many forms of flexible couplings are used to connect shafts that may not be exactly in line. Some of these depend on the flexibility of fabric, rubber, or leather; while others are of all-metal construction.

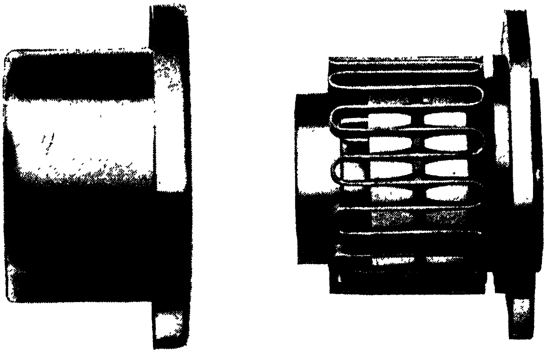
A simple flexible coupling consists of two flanges, one of which is provided with projecting pins extending into rubber-bushed holes in the mating flange.



Smith and Serrell, Newark, N. J.

FIG. 287. FRANCKE FLEXIBLE COUPLING.

In the Francke coupling, Fig. 287, the pins and rubber bushings have been replaced by steel pins made up of bundles of thin spring steel which provide the necessary flexibility. The pins are held in one flange and can slide in the other thus providing for axial movement of the shafts.



The Falk Corporation, Milwaukee, Wis.

FIG. 288. FALK FLEXIBLE COUPLING.

Another type of flexible coupling which makes use of a bent steel spring, or grid-spring, fitting into grooves cut in the hubs, is shown in Fig. 288.

The flanges of the flexible coupling in Fig. 289 are connected

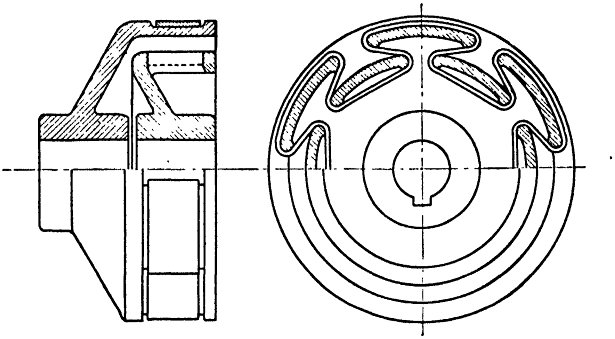
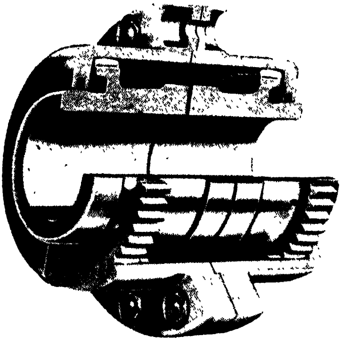


FIG. 289

by an endless belt (sometimes of rope) interlaced between pins or other supports, projecting from the mating flanges, the belt providing the necessary flexibility.

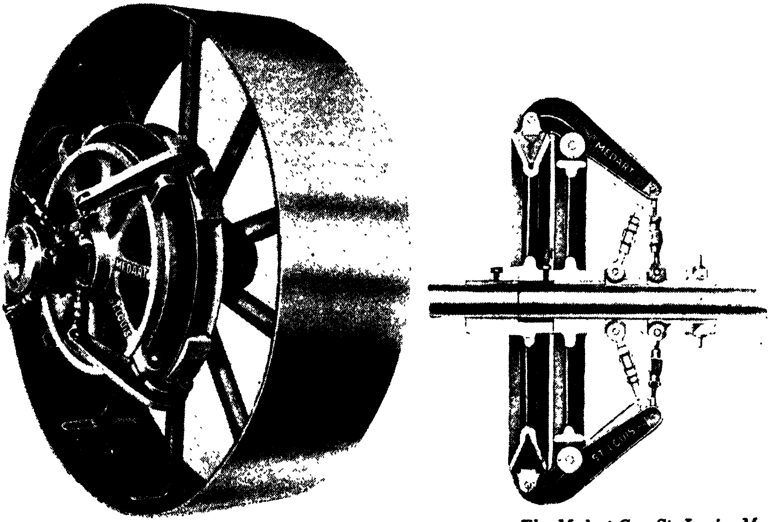
An all-metal gear-type flexible coupling is shown in Fig. 290. The gear teeth on the hubs engage with gear teeth cut on the inside of the housing. The crown surface of the hub teeth are spherically formed to permit self-alignment without binding.



Poole Engineering and Machine Co., Baltimore, Md.

FIG. 290. GEAR-TYPE FLEXIBLE COUPLING.

269. Clutches. When it is necessary to provide a means of connecting a pulley and its shaft or two shafts in such a way that they can be connected or disconnected at will, some type of clutch is used. Figures 291 and 292 show common types of friction clutches with the linkwork used to operate them. The power is transmitted by friction between an iron drum and blocks of wood, molded asbestos, or other material having a high coefficient of friction.



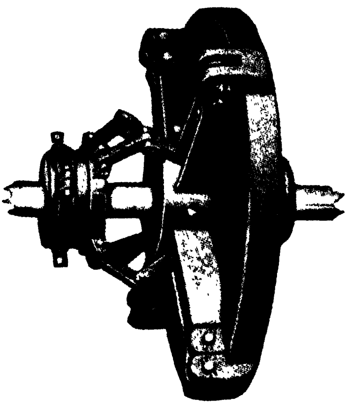
The Medart Co., St. Louis, Mo.

(a) (b)

FIG. 291. V-BLOCK FRICTION CLUTCH.

270. Straight-Line Motions. Straight-line motions are mechanisms in which one point is made to move in a straight line. They may depend on guides or cams acting as the guiding device, but the most common are linkwork. Some produce exact straight-line motion, while others produce only approximate straight-line motion.

In the early days of the development of the steam engine, straight-line motions were used to guide the end of the piston rod, as a substitute for the present-day cross-head and guides. They are not used for this purpose at present.



Stephens-Adamson Mfg. Co., Aurora, Ill.

FIG. 292. STEEL PLATE FRICTION CLUTCH.

271. Peaucellier's Straight-Line Motion. This is probably the most familiar of the exact straight-line mechanisms. In Fig. 293, the links AD and AE oscillate on the fixed center A. The

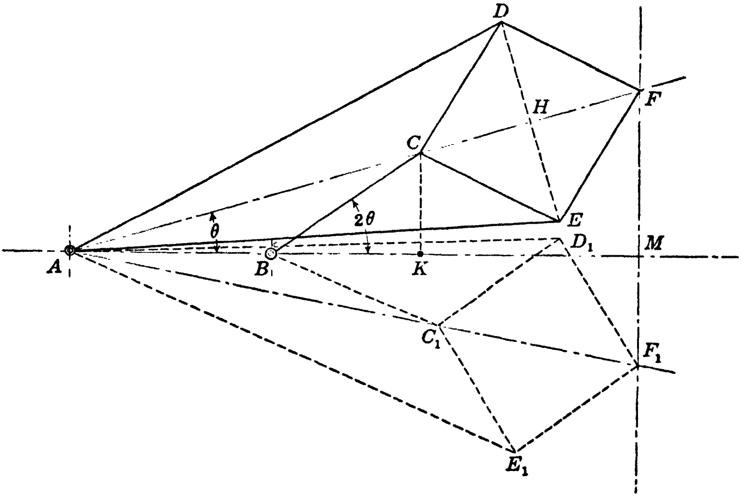


FIG. 293

links CD, CE, DF, and EF are equal. The link BC oscillating on the fixed center B has a length equal to the center distance AB.

From C and F draw perpendiculars to the line of centers, AB, extended. From the figure, we find

$$\cos\theta = \frac{AK}{AC} = \frac{AM}{AF}, \qquad \cos^2\theta = \frac{AK}{AC} \times \frac{AM}{AF},$$

$$AM = \frac{AC \times AF \times \cos^2\theta}{AK} \tag{110}$$

$$= \frac{(AH - CH)\,(AH + CH)\cos^2\theta}{AB + BC\cos 2\theta} \tag{111}$$

$$= \frac{(AH^2 - CH^2)\cos^2\theta}{AB(1 + \cos 2\theta)} \tag{112}$$

$$= \frac{\{(AD^2 - HD^2) - (CD^2 - HD^2)\}\cos^2\theta}{AB(1 + \cos 2\theta)} \tag{113}$$

or

$$AM = \frac{(AD^2 - CD^2)}{AB} \times \frac{\cos^2\theta}{2\cos^2\theta} = \frac{AD^2 - CD^2}{2AB}. \tag{114}$$

Therefore AM is a constant. It follows that the point F will always be on a line drawn through M perpendicular to ABM, and therefore F moves in a straight line.

272. The Isosceles Linkage. The isosceles linkage is a modification of the crank-and-cross-head mechanism in which the crank is pinned to the mid-point of a connecting rod twice as long as the crank. Figure 294 shows this linkage. Since AB is equal to BC equal to BD, it is evident that, if C moves in a straight line through the center A, D will move in a straight line through A and perpendicular to AC. This is known as the ***Scott-Russell motion.*** If CD is extended to T, as in Fig. 294, T will trace an ellipse whose major axis is 2 × CT and whose minor axis is 2 × DT. This combination is called the ***Ellipsograph.***

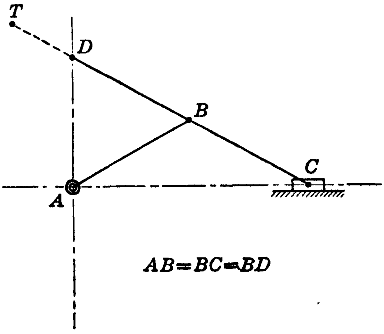


FIG. 294

273. The Grasshopper or Modified Scott-Russell Linkage. This linkage, shown in Fig. 295, is a modification of the isosceles linkage, a long link *EC* being substituted for the sliding member. If the link *CE* is of considerable length, and *C* moves only through short distances from the position where *CE* is perpendicular to *AC*, the point *T* will move in a path very closely approximating a straight line.

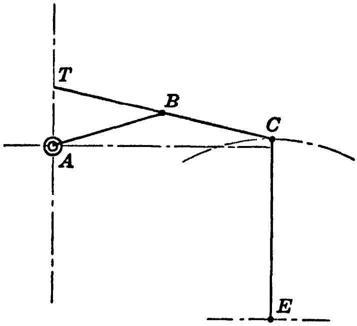


Fig. 295

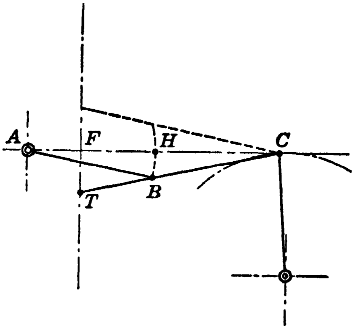


Fig. 296

It is usually inconvenient to have the tracing point *T* move on a line passing through the fixed point *A*. The point *B* is then chosen so that *CB* is greater than *BT*, as in Fig. 296. The link *CT* is placed with *T* at the extremity of its travel and *TF* drawn perpendicular to *FC*. When *T* is at *F*, the point *B* will be at *H*. The point *A* located on *CFA* is the center of a circle passing through *B* and *H*. The deviation of the path of *T* from a straight line is very small, but increases as *B* moves away from the midpoint of *CT*.

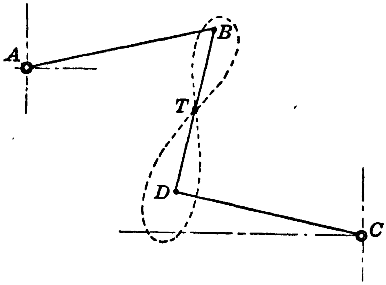


Fig. 297

274. Watt's Approximate Straight-Line Linkage. Watt's straight-line linkage, shown in Fig. 297, consists of two links, *AB* and *CD*, fixed at *A* and *C* and connected by link *BD*. As *AB* and *CD* oscillate, a point, *T*, on the link *BD* will trace a figure-eight path, portions of which are nearly straight lines. The point, T, is located so that

$$\frac{BT}{DT} = \frac{CD}{AB}.$$

If AB and CD are made equal, then T is the mid-point of the link.

Consider the crank AB to be moved through the angle θ to the position AB_1 in Fig. 298. Point B has moved to the left while D has moved to the right, so that the link BD assumes the position B_1D_1, crossing the vertical line BD extended at T_1. The point T may be used as a tracing point, which will be on the line BD, extended, in the two positions shown, and will not deviate greatly from this line during the movement of the links. The location of T may be found from the figure as follows:

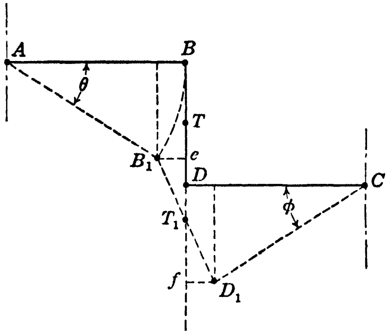


FIG. 298

$$\frac{B_1T_1}{D_1T_1} = \frac{B_1e}{D_1f} = \frac{AB - AB_1\cos\theta}{CD - CD_1\cos\phi} = \frac{AB(1 - \cos\theta)}{CD(1 - \cos\phi)};$$

but we know that

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 1 - 2\sin^2\frac{\theta}{2}, \qquad \cos\phi = 1 - 2\sin^2\frac{\phi}{2}.$$

Hence we may write

$$\frac{B_1T_1}{D_1T_1} = \frac{AB\sin^2\frac{\theta}{2}}{CD\sin^2\frac{\phi}{2}} = \frac{CD\left(AB\sin\frac{\theta}{2}\right)^2}{AB\left(CD\sin\frac{\phi}{2}\right)^2} = \frac{CD}{AB}\frac{(BB_1)^2}{(D_1D)^2}.$$

For small movements, the chord BB_1 is practically equal to DD_1. In practice, the angle θ has a maximum value of 15° to 20° and BB_1/DD_1 can be taken as unity without appreciable error. Then we have

$$\frac{BT}{DT} = \frac{B_1T_1}{D_1T_1} = \frac{CD}{AB}, \tag{115}$$

which locates the tracing point T on the link BD so that the segments of BD are inversely proportional to the adjacent links.

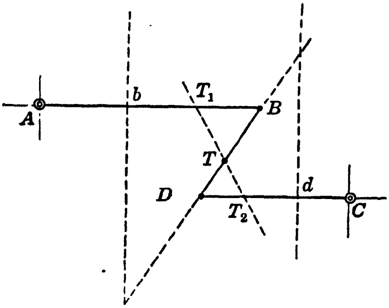
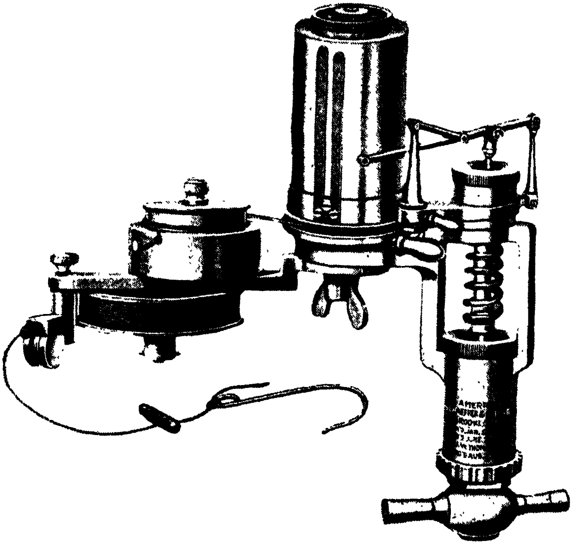


FIG. 299

Point T does not travel on a straight line from T to T_1, but the deviation is small. The deviation can be decreased if the lengths of the links AB and CD are increased, as in Fig. 299.



Consolidated Ashcroft Hancock Co., Inc., Bridgeport, Conn.

FIG. 300. AMERICAN-THOMPSON IMPROVED ENGINE-INDICATOR.

In the mid-position of the links, *AB* and *CD* should be parallel. The tracing point *T* is located if a line T_1TT_2 is drawn, which bisects the lines *Bb* and *Dd* as shown.

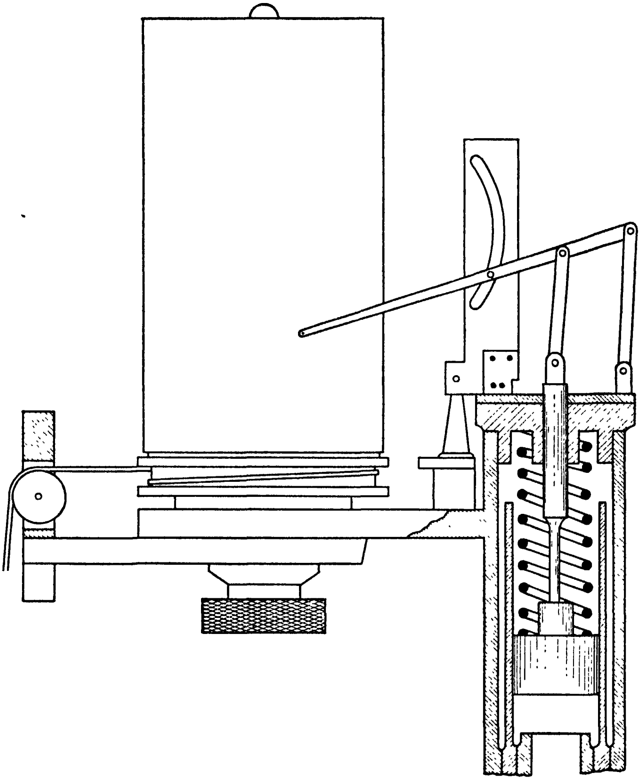
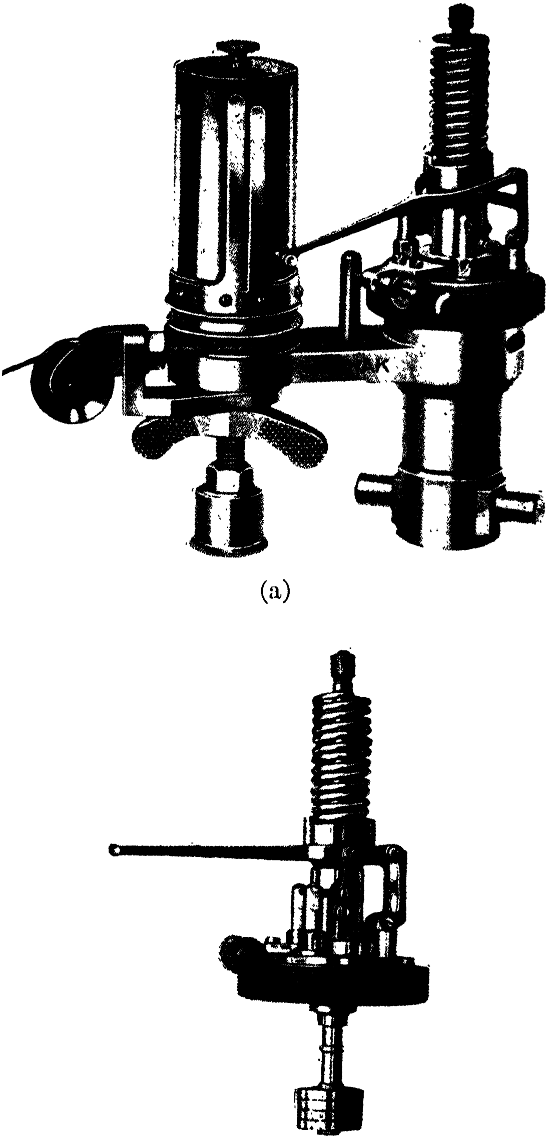


Fig. 301

275. Indicator Motions. Probably the most important of the straight-line motions are those used on engine indicators. The function of the engine-indicator linkage is to multiply the motion of the indicator piston, causing a tracing point to move in a straight line through distances directly proportional to the piston movements. The linkages must be light in construction in order to decrease inertia effects. They are all applications of the straight-line motions described in the previous paragraphs, and



(a)

(b)

Bacharach Industrial Instrument Co., Pittsburgh, Pa.

FIG. 302. MAIHAK ENGINE-INDICATOR.

the following examples will illustrate some of the common types in use.

The American Thompson indicator, Fig. 300, shows an application of the grasshopper linkage, and the Tabor indicator, Fig. 301, uses a similar linkage in which a roller and curved guiding-slot replaces one of the links. The Maihak indicator, Fig. 302, shows an application of the modified pantograph linkage.

There are other indicators on the market which make use of some of the other straight-line motions, and the student should study the linkages used by other manufacturers.

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